Turkey Team Selection Test 1993

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by GlassBead

1 Show that there exists an infinite arithmetic progression of natural numbers such that the first term is 16 and the number of positive divisors of each term is divisible by 5. Of all such sequences, find the one with the smallest possible common difference.

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2 Let M be the circumcenter of an acute-angled triangle ABC. The circumcircle of triangle BMA intersects BC at P and AC at Q. Show that $CM \perp PQ$.

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3 Let (b_n) be a sequence such that $b_n \geq 0$ and $b_{n+1}^2 \geq \frac{b_1^2}{1^3} + \cdots + \frac{b_n^2}{n^3}$ for all $n \geq 1$. Prove that there exists a natural number K such that

$$\sum_{n=1}^{K} \frac{b_{n+1}}{b_1 + b_2 + \cdots + b_n} \geq \frac{1993}{1000}$$

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4 Some towns are connected by roads, with at most one road between any two towns. Let v be the number of towns and e be the number of roads. Prove that

(a) if $e < v - 1$, then there are two towns such that one cannot travel between them;
(b) if $2e > (v-1)(v-2)$, then one can travel between any two towns.

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5 Points E and C are chosen on a semicircle with diameter AB and center O such that $OE \perp AB$ and the intersection point D of AC and OE is inside the semicircle. Find all values of $\angle CAB$ for which the quadrilateral $OBCD$ is tangent.

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6 Determine all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ that satisfy:

$$f\left(x + \frac{y}{x}\right) = f(x) + f\left(\frac{y}{x}\right) + 2y \text{ for all } x, y \in \mathbb{Q}^+$$
