Turkey Team Selection Test 1999

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Day 1

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1 Let $m \leq n$ be positive integers and p be a prime. Let $p-$expansions of m and n be

$$m = a_0 + a_1 p + \dots + a_r p^r$$

$$n = b_0 + b_1 p + \dots + b_s p^s$$

respectively, where $a_r, b_s \neq 0$, for all $i \in \{0,1,\dots,r\}$ and for all $j \in \{0,1,\dots,s\}$, we have $0 \leq a_i, b_j \leq p-1$.

If $a_i \leq b_i$ for all $i \in \{0,1,\dots,r\}$, we write $m \prec_p n$. Prove that

$$p \nmid \binom{n}{m} \Leftrightarrow m \prec_p n$$

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2 Let L and N be the mid-points of the diagonals $[AC]$ and $[BD]$ of the cyclic quadrilateral $ABCD$, respectively. If BD is the bisector of the angle ANC, then prove that AC is the bisector of the angle BLD.

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3 Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that the set

$$\left\{ \frac{f(x)}{x} : x \neq 0 \text{ and } x \in \mathbb{R} \right\}$$

is finite, and for all $x \in \mathbb{R}$

$$f(x - 1 - f(x)) = f(x) - x - 1$$

Day 2

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1 Let the area and the perimeter of a cyclic quadrilateral C be A_C and P_C, respectively. If the area and the perimeter of the quadrilateral which is tangent to the circumcircle of C at the vertices of C are A_T and P_T , respectively, prove that $\frac{A_C}{A_T} \geq \left(\frac{P_C}{P_T}\right)^2$.

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2 Each of A, B, C, D, E, and F knows a piece of gossip. They communicate by telephone via a central switchboard, which can connect only two of them at a time. During a conversation, each side tells the other everything he or she knows at that point. Determine the minimum number of calls for everyone to know all six pieces of gossip.

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3 Prove that the plane is not a union of the inner regions of finitely many parabolas. (The outer region of a parabola is the union of the lines not intersecting the parabola. The inner region of a parabola is the set of points of the plane that do not belong to the outer region of the parabola)
