Turkey Team Selection Test 2000

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by Potla

Day 1

1 (a) Prove that for every positive integer n, the number of ordered pairs (x, y) of integers satisfying $x^2 - xy + y^2 = n$ is divisible by 3. (b) Find all ordered pairs of integers satisfying $x^2 - xy + y^2 = 727$.

2 In a triangle ABC, the internal and external bisectors of the angle A intersect the line BC at D and E respectively. The line AC meets the circle with diameter DE again at F. The tangent line to the circle ABF at A meets the circle with diameter DE again at G. Show that $AF = AG$.

3 Let $P(x) = x + 1$ and $Q(x) = x^2 + 1$. Consider all sequences $\langle (x_k, y_k) \rangle_{k \in \mathbb{N}}$ such that $(x_1, y_1) = (1, 3)$ and (x_{k+1}, y_{k+1}) is either $(P(x_k), Q(y_k))$ or $(Q(x_k), P(y_k))$ for each k. We say that a positive integer n is nice if $x_n = y_n$ holds in at least one of these sequences. Find all nice numbers.

Day 2

1 Show that any triangular prism of innite length can be cut by a plane such that the resulting intersection is an equilateral triangle.

2 Points M, N, K, L are taken on the sides AB, BC, CD, DA of a rhombus $ABCD$, respectively, in such a way that $MN \parallel LK$ and the distance between MN and KL is equal to the height of $ABCD$. Show that the circumcircles of the triangles ALM and NCK intersect each other, while those of LDK and MBN do not.

3 Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function such that

$$|f(x + y) - f(x) - f(y)| \le 1 \quad \text{for all} \quad x, y \in \mathbb{R}.$$

Prove that there is a function $g : \mathbb{R} \to \mathbb{R}$ such that $|f(x) - g(x)| \le 1$ and $g(x + y) = g(x) + g(y)$ for all $x, y \in \mathbb{R}$.