Turkey Team Selection Test 2003

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Day 1

1 Let $M=\{(a,b,c,d)|a,b,c,d\in\{1,2,3,4\}$ and $abcd>1\}$. For each $n\in\{1,2,\dots,254\}$, the sequence (a_1,b_1,c_1,d_1), (a_2,b_2,c_2,d_2), $\dots$, $(a_{255},b_{255},c_{255},d_{255})$ contains each element of M exactly once and the equality

$$|a_{n+1}-a_n|+|b_{n+1}-b_n|+|c_{n+1}-c_n|+|d_{n+1}-d_n|=1$$

holds. If $c_1=d_1=1$, find all possible values of the pair (a_1,b_1).

2 Let K be the intersection of the diagonals of a convex quadrilateral $ABCD$. Let $L\in[AD]$, $M\in[AC]$, $N\in[BC]$ such that $KL\parallel AB$, $LM\parallel DC$, $MN\parallel AB$. Show that

$$\frac{Area(KLMN)}{Area(ABCD)}<\frac{8}{27}.$$

3 Is there an arithmetic sequence with

a. 2003

b. infinitely many

terms such that each term is a power of a natural number with a degree greater than 1?

Day 2

4 Find the least

a. positive real number

b. positive integer

t such that the equation $(x^2+y^2)^2+2tx(x^2+y^2)=t^2y^2$ has a solution where x,y are positive integers.

5 Let A be a point on a circle with center O and B be the midpoint of $[OA]$. Let C and D be points on the circle such that they lie on the same side of the line OA and $\widehat{CBO}=\widehat{DBA}$. Show that the reflection of the midpoint of $[CD]$ over B lies on the circle.

6 For all positive integers n, let $p(n)$ be the number of non-decreasing sequences of positive integers such that for each sequence, the sum of all terms of the sequence is equal to n. Prove that
$$\frac{1+p(1)+p(2)+\cdots+p(n-1)}{p(n)} \leq \sqrt{2n}.$$