Turkey Team Selection Test 2004

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by xeroxia

Day 1

1 An 11×11 chess board is covered with one $\square$ shaped and forty $\square\square\square$ shaped tiles. Determine the squares where $\square$ shaped tile can be placed.

2 Show that

$$\min\{|PA|,|PB|,|PC|\} + |PA| + |PB| + |PC| < |AB| + |BC| + |CA|$$

if P is a point inside $\triangle ABC$.

3 Let n be a positive integer. Determine integers, $n+1 \leq r \leq 3n+2$ such that for all integers $a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_m$ satisfying the equations

$$a_1 b_1^k + a_2 b_2^k + \cdots + a_m b_m^k = 0$$

for every $1 \leq k \leq n$, the condition

$$r \mid a_1 b_1^r + a_2 b_2^r + \cdots + a_m b_m^r = 0$$

also holds.

Day 2

1 Find all possible values of $x - \lfloor x \rfloor$ if $\sin \alpha = 3/5$ and $x = 5^{2003} \sin(2004\alpha)$.

2 Let $\triangle ABC$ be an acute triangle, O be its circumcenter, and D be a point different that A and C on the smaller AC arc of its circumcircle. Let P be a point on $[AB]$ satisfying $\widehat{ADP} = \widehat{OBC}$ and Q be a point on $[BC]$ satisfying $\widehat{CDQ} = \widehat{OBA}$. Show that $\widehat{DPQ} = \widehat{DOC}$.

3 Each student in a classroom has $0, 1, 2, 3, 4, 5$ or 6 pieces of candy. At each step the teacher chooses some of the students, and gives one piece of candy to each of them and also to any other student in the classroom who is friends with at least one of these students. A student who receives the seventh piece eats all 7 pieces. Assume that for every pair of students in the classroom, there is at least one student who is friend swith exactly one of them. Show that no matter how many pieces each student has at the beginning, the teacher can make them to have any desired numbers of pieces after finitely many steps.