Turkey Team Selection Test 2006

www.artofproblemsolving.com/community/c5461

by Valentin Vornicu, silouan

Day 1

1 Find the maximum value for the area of a heptagon with all vertices on a circle and two diagonals perpendicular.

2 How many ways are there to divide a $2 \times n$ rectangle into rectangles having integral sides, where n is a positive integer?

3 If x, y, z are positive real numbers and $xy + yz + zx = 1$ prove that

$$\frac{27}{4}(x+y)(y+z)(z+x) \geq (\sqrt{x+y} + \sqrt{y+z} + \sqrt{z+x})^2 \geq 6\sqrt{3}.$$

Day 2

1 For all integers $n \geq 1$ we define $x_{n+1} = x_1^2 + x_2^2 + \cdots + x_n^2$, where x_1 is a positive integer. Find the least x_1 such that 2006 divides x_{2006}.

2 From a point Q on a circle with diameter AB different from A and B, we draw a perpendicular to AB, QH, where H lies on AB. The intersection points of the circle of diameter AB and the circle of center Q and radius QH are C and D. Prove that CD bisects QH.

3 Each one of 2006 students makes a list with 12 schools among 2006. If we take any 6 students, there are two schools which at least one of them is included in each of 6 lists. A list which includes at least one school from all lists is a good list.

a) Prove that we can always find a good list with 12 elements, whatever the lists are;

b) Prove that students can make lists such that no shorter list is good.