Turkey Team Selection Test 2008

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Day 1

1 In an ABC triangle such that $m(\angle B) > m(\angle C)$, the internal and external bisectors of vertice A intersects BC respectively at points D and E. P is a variable point on EA such that A is on $[EP]$. DP intersects AC at M and ME intersects AD at Q. Prove that all PQ lines have a common point as P varies.

2 A graph has 30 vertices, 105 edges and 4822 unordered edge pairs whose endpoints are disjoint. Find the maximal possible difference of degrees of two vertices in this graph.

3 The equation $x^3 - ax^2 + bx - c = 0$ has three (not necessarily different) positive real roots. Find the minimal possible value of $\frac{1+a+b+c}{3+2a+b} - \frac{c}{b}$.

Day 2

4 The sequence (x_n) is defined as; $x_1 = a, x_2 = b$ and for all positive integer n, $x_{n+2} = 2008x_{n+1} - x_n$. Prove that there are some positive integers a, b such that $1 + 2006x_{n+1}x_n$ is a perfect square for all positive integer n.

5 D is a point on the edge BC of triangle ABC such that $AD = \frac{BD^2}{AB+AD} = \frac{CD^2}{AC+AD}$. E is a point such that D is on $[AE]$ and $CD = \frac{DE^2}{CD+CE}$. Prove that $AE = AB + AC$.

6 There are n voters and m candidates. Every voter makes a certain arrangement list of all candidates (there is one person in every place $1, 2, ...m$) and votes for the first k people in his/her list. The candidates with most votes are selected and say them winners. A poll profile is all of this n lists.
If a is a candidate, R and R' are two poll profiles. R' is $a-good$ for R if and only if for every voter; the people which in a worse position than a in R is also in a worse position than a in R'. We say positive integer k is monotone if and only if for every R poll profile and every winner a for R poll profile is also a winner for all $a-good$ R' poll profiles. Prove that k is monotone if and only if $k > \frac{m(n-1)}{n}$.