Turkey Team Selection Test 2009

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by orl

Day 1

1 Find all $f : Q^+ \to Z$ functions that satisfy $f\left(\frac{1}{x}\right) = f(x)$ and $(x+1)f(x-1) = xf(x)$ for all rational numbers that are bigger than 1.

2 Quadrilateral $ABCD$ has an inscribed circle which centered at O with radius r. AB intersects CD at P; AD intersects BC at Q and the diagonals AC and BD intersects each other at K. If the distance from O to the line PQ is k, prove that $OK \cdot k = r^2$.

3 Within a group of 2009 people, every two people has exactly one common friend. Find the least value of the difference between the person with maximum number of friends and the person with minimum number of friends.

Day 2

1 For which p prime numbers, there is an integer root of the polynominal $1+p+Q(x^1)\cdot Q(x^2)\ldots Q(x^{2p-2})$ such that $Q(x)$ is a polynominal with integer coefficients?

2 In a triangle ABC incircle touches the sides AB, AC and BC at C_1, B_1 and A_1 respectively. Prove that $\sqrt{\frac{AB_1}{AB}} + \sqrt{\frac{BC_1}{BC}} + \sqrt{\frac{CA_1}{CA}} \leq \frac{3}{\sqrt{2}}$ is true.

3 In a class of $n \geq 4$ some students are friends. In this class any $n-1$ students can be seated in a round table such that every student is sitting next to a friend of him in both sides, but n students can not be seated in that way. Prove that the minimum value of n is 10.