Turkey Team Selection Test 2010

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Day 1 March 27th

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1 $D,\ E,\ F$ are points on the sides $AB,\ BC,\ CA,$ respectively, of a triangle ABC such that $AD=AF,\ BD=BE,$ and $DE=DF.$ Let I be the incenter of the triangle $ABC,$ and let K be the point of intersection of the line BI and the tangent line through A to the circumcircle of the triangle $ABI.$ Show that $AK=EK$ if $AK=AD.$

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2 Show that

$$\sum_{cyc} \sqrt[4]{\frac{(a^2+b^2)(a^2-ab+b^2)}{2}} \leq \frac{2}{3}(a^2+b^2+c^2)\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right)$$

for all positive real numbers $a,\ b,\ c.$

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3 A teacher wants to divide the 2010 questions she asked in the exams during the school year into three folders of 670 questions and give each folder to a student who solved all 670 questions in that folder. Determine the minimum number of students in the class that makes this possible for all possible situations in which there are at most two students who did not solve any given question.

Day 2 March 28th

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1 Let $0 \leq k < n$ be integers and $A=\{a \ : \ a \equiv k \pmod n \}.$ Find the smallest value of n for which the expression

$$\frac{a^m+3^m}{a^2-3a+1}$$

does not take any integer values for $(a,m) \in A \times \mathbb{Z}^+.$

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2 For an interior point D of a triangle $ABC,$ let Γ_D denote the circle passing through the points $A,\ E,\ D,\ F$ if these points are concyclic where $BD \cap AC=\{E\}$ and $CD \cap AB=\{F\}.$ Show that all circles Γ_D pass through a second common point different from A as D varies.

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3 Let Λ be the set of points in the plane whose coordinates are integers and let F be the collection of all functions from Λ to $\{1,-1\}.$ We call a function f in F *perfect* if every function g in F that differs from f at finitely many points satisfies the condition

$$\sum_{0<d(P,Q)<2010} \frac{f(P)f(Q)-g(P)g(Q)}{d(P,Q)} \geq 0$$

where $d(P, Q)$ denotes the distance between P and Q. Show that there exist infinitely many *perfect* functions that are not translates of each other.
