Turkey Team Selection Test 2011
www.artofproblemsolving.com/community/c5466
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Day 1

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1 Let $\mathbb{Q}^+$ denote the set of positive rational numbers. Determine all functions $f:\mathbb{Q}^+\to\mathbb{Q}^+$ that satisfy the conditions

$$f\left(\frac{x}{x+1}\right)=\frac{f(x)}{x+1} \qquad \text{and} \qquad f\left(\frac{1}{x}\right)=\frac{f(x)}{x^3}$$

for all $x\in\mathbb{Q}^+$.

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2 Let I be the incenter and AD be a diameter of the circumcircle of a triangle ABC. If the point E on the ray BA and the point F on the ray CA satisfy the condition

$$BE=CF=\frac{AB+BC+CA}{2}$$

show that the lines EF and DI are perpendicular.

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3 Let A and B be sets with 2011^2 and 2010 elements, respectively. Show that there is a function $f:A\times A\to B$ satisfying the condition $f(x,y)=f(y,x)$ for all $(x,y)\in A\times A$ such that for every function $g:A\to B$ there exists $(a_1,a_2)\in A\times A$ with $g(a_1)=f(a_1,a_2)=g(a_2)$ and $a_1\neq a_2$.

Day 2

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1 Let D be a point different from the vertices on the side BC of a triangle ABC. Let $I,\ I_1$ and I_2 be the incenters of the triangles $ABC,\ ABD$ and ADC, respectively. Let E be the second intersection point of the circumcircles of the triangles AI_1I and ADI_2, and F be the second intersection point of the circumcircles of the triangles AII_2 and AI_1D. Prove that if $AI_1=AI_2$, then

$$\frac{EI}{FI}\cdot\frac{ED}{FD}=\frac{{EI_1}^2}{{FI_1}^2}.$$

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2 Let a,b,c be positive real numbers satisfying $a^2+b^2+c^2\geq 3$. Prove that

$$\frac{(a+1)(b+2)}{(b+1)(b+5)}+\frac{(b+1)(c+2)}{(c+1)(c+5)}+\frac{(c+1)(a+2)}{(a+1)(a+5)}\geq\frac{3}{2}$$

3 Let $t(n)$ be the sum of the digits in the binary representation of a positive integer $n,$ and let $k \geq 2$ be an integer.

a. Show that there exists a sequence $(a_i)_{i=1}^{\infty}$ of integers such that $a_m \geq 3$ is an odd integer and $t(a_1 a_2 \cdots a_m) = k$ for all $m \geq 1.$

b. Show that there is an integer N such that $t(3 \cdot 5 \cdots (2m+1)) > k$ for all integers $m \geq N.$

Day 3

1 Let K be a point in the interior of an acute triangle ABC and $ARBPCQ$ be a convex hexagon whose vertices lie on the circumcircle Γ of the triangle $ABC.$ Let A_1 be the second point where the circle passing through K and tangent to Γ at A intersects the line $AP.$ The points B_1 and C_1 are defined similarly. Prove that

$$\min \left\{ \frac{PA_1}{AA_1}, \frac{QB_1}{BB_1}, \frac{RC_1}{CC_1} \right\} \leq 1.$$

2 Graphistan has 2011 cities and Graph Air (GA) is running one-way flights between all pairs of these cities. Determine the maximum possible value of the integer k such that no matter how these flights are arranged it is possible to travel between any two cities in Graphistan riding only GA flights as long as the absolute values of the difference between the number of flights originating and terminating at any city is not more than $k.$

3 Let p be a prime, n be a positive integer, and let $\mathbb{Z}_{p^n}$ denote the set of congruence classes modulo p^n. Determine the number of functions $f : \mathbb{Z}_{p^n} \to \mathbb{Z}_{p^n}$ satisfying the condition

$$f(a) + f(b) \equiv f(a + b + pab) \pmod{p^n}$$

for all $a, b \in \mathbb{Z}_{p^n}.$