Turkey Team Selection Test 2012

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Day 1

1 Let $A=\{1,2,\ldots,2012\}$, $B=\{1,2,\ldots,19\}$ and S be the set of all subsets of A. Find the number of functions $f:S\to B$ satisfying $f(A_1\cap A_2)=\min\{f(A_1),f(A_2)\}$ for all $A_1,A_2\in S$.

2 In an acute triangle ABC, let D be a point on the side BC. Let M_1,M_2,M_3,M_4,M_5 be the midpoints of the line segments AD,AB,AC,BD,CD, respectively and O_1,O_2,O_3,O_4 be the circumcenters of triangles $ABD,ACD,M_1M_2M_4,M_1M_3M_5$, respectively. If S and T are midpoints of the line segments AO_1 and AO_2, respectively, prove that SO_3O_4T is an isosceles trapezoid.

3 For all positive real numbers a,b,c satisfying $ab+bc+ca\leq 1$, prove that

$$a+b+c+\sqrt{3}\geq 8abc\left(\frac{1}{a^2+1}+\frac{1}{b^2+1}+\frac{1}{c^2+1}\right)$$

Day 2

1 In a triangle ABC, incircle touches the sides BC,CA,AB at D,E,F, respectively. A circle ω passing through A and tangent to line BC at D intersects the line segments BF and CE at K and L, respectively. The line passing through E and parallel to DL intersects the line passing through F and parallel to DK at P. If R_1,R_2,R_3,R_4 denotes the circumradius of the triangles AFD,AED,FPD,EPD, respectively, prove that $R_1R_4=R_2R_3$.

2 A positive integer n is called *good* if for all positive integers a which can be written as $a=n^2\sum_{i=1}^{n}{x_i}^2$ where $x_1,x_2,\ldots,x_n$ are integers, it is possible to express a as $a=\sum_{i=1}^{n}{y_i}^2$ where $y_1,y_2,\ldots,y_n$ are integers with none of them is divisible by n. Find all good numbers.

3 Two players A and B play a game on a $1\times m$ board, using 2012 pieces numbered from 1 to 2012. At each turn, A chooses a piece and B places it to an empty place. After k turns, if all pieces are placed on the board increasingly, then B wins, otherwise A wins. For which values of (m,k) pairs can B guarantee to win?

Day 3

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1 Let $S_r(n) = 1^r + 2^r + \cdots + n^r$ where n is a positive integer and r is a rational number. If $S_a(n) = (S_b(n))^c$ for all positive integers n where a, b are positive rationals and c is positive integer then we call (a, b, c) as *nice triple.* Find all nice triples.

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2 In a plane, the six different points A, B, C, A', B', C' are given such that triangles ABC and $A'B'C'$ are congruent, i.e. $AB = A'B', BC = B'C', CA = C'A'$. Let G be the centroid of ABC and A_1 be an intersection point of the circle with diameter AA' and the circle with center A' and passing through G. Define B_1 and C_1 similarly. Prove that

$$AA_1^2 + BB_1^2 + CC_1^2 \leq AB^2 + BC^2 + CA^2$$

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3 Let $\mathbb{Z}^+$ and $\mathbb{P}$ denote the set of positive integers and the set of prime numbers, respectively. A set A is called $S-$proper where $A, S \subset \mathbb{Z}^+$ if there exists a positive integer N such that for all $a \in A$ and for all $0 \leq b < a$ there exist $s_1, s_2, \dots, s_n \in S$ satisfying $b \equiv s_1 + s_2 + \cdots + s_n \pmod a$ and $1 \leq n \leq N$.
Find a subset S of $\mathbb{Z}^+$ for which $\mathbb{P}$ is $S-$proper but $\mathbb{Z}^+$ is not.
