Turkey Team Selection Test 2013

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by xeroxia

Day 1 March 30th

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1 Let $\phi(n)$ be the number of positive integers less than n that are relatively prime to n, where n is a positive integer. Find all pairs of positive integers (m,n) such that

$$2^n + (n-\phi(n)-1)! = n^m+1.$$

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2 We put pebbles on some unit squares of a 2013×2013 chessboard such that every unit square contains at most one pebble. Determine the minimum number of pebbles on the chessboard, if each 19×19 square formed by unit squares contains at least 21 pebbles.

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3 Let O be the circumcenter and I be the incenter of an acute triangle ABC with $m(\widehat{B}) \neq m(\widehat{C})$. Let D, E, F be the midpoints of the sides $[BC]$, $[CA]$, $[AB]$, respectively. Let T be the foot of perpendicular from I to $[AB]$. Let P be the circumcenter of the triangle DEF and Q be the midpoint of $[OI]$. If A, P, Q are collinear, prove that

$$\frac{|AO|}{|OD|}-\frac{|BC|}{|AT|}=4.$$

Day 2 March 31st

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1 Find all pairs of integers (m,n) such that $m^6 = n^{n+1} + n - 1$.

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2 Let the incircle of the triangle ABC touch $[BC]$ at D and I be the incenter of the triangle. Let T be midpoint of $[ID]$. Let the perpendicular from I to AD meet AB and AC at K and L, respectively. Let the perpendicular from T to AD meet AB and AC at M and N, respectively. Show that $|KM|\cdot|LN| = |BM|\cdot|CN|$.

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3 For all real numbers x,y,z such that $-2\le x,y,z\le 2$ and $x^2+y^2+z^2+xyz=4$, determine the least real number K satisfying

$$\frac{z(xz+yz+y)}{xy+y^2+z^2+1}\le K.$$

Day 3 April 1st

1 Let E be intersection of the diagonals of convex quadrilateral $ABCD$. It is given that $m(\widehat{EDC}) = m(\widehat{DEC}) = m(\widehat{BAD})$. If F is a point on $[BC]$ such that $m(\widehat{BAF}) + m(\widehat{EBF}) = m(\widehat{BFE})$, show that A, B, F, D are concyclic.

2 Determine all functions $f : \mathbf{R} \to \mathbf{R}^+$ such that for all real numbers x, y the following conditions hold:

$$\begin{array}{rl} i. & f(x^2) = f(x)^2 - 2xf(x) \\ ii. & f(-x) = f(x-1) \\ iii. & 1 < x < y \Longrightarrow f(x) < f(y). \end{array}$$

3 Some cities of a country consisting of n cities are connected by round trip flights so that there are at least k flights from any city and any city is reachable from any city. Prove that for any such flight organization these flights can be distributed among $n-k$ air companies so that one can reach any city from any city by using of at most one flight of each air company.