Turkey Team Selection Test 2014
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Day 1 March 1st

1 Find the number of $(a_1, a_2, ..., a_{2014})$ permutations of the $(1, 2, ..., 2014)$ such that, for all $1 \leq i < j \leq 2014$, $i + a_i \leq j + a_j$.

2 Find all f functions from real numbers to itself such that for all real numbers x, y the equation

$$f(f(y) + x^2 + 1) + 2x = y + (f(x + 1))^2$$

holds.

3 Let r, R and r_a be the radii of the incircle, circumcircle and A-excircle of the triangle ABC with $AC > AB$, respectively. I, O and J_A are the centers of these circles, respectively. Let incircle touches the BC at D, for a point $E \in (BD)$ the condition $A(IEJ_A) = 2A(IEO)$ holds.
Prove that

$$ED = AC - AB \iff R = 2r + r_a.$$

Day 2 March 2nd

1 Find all pairs (m, n) of positive odd integers, such that $n \mid 3m + 1$ and $m \mid n^2 + 3$.

2 A circle ω cuts the sides BC, CA, AB of the triangle ABC at A_1 and A_2; B_1 and B_2; C_1 and C_2, respectively. Let P be the center of ω. A' is the circumcenter of the triangle A_1A_2P, B' is the circumcenter of the triangle B_1B_2P, C' is the circumcenter of the triangle C_1C_2P. Prove that AA', BB' and CC' concur.

3 Prove that for all all non-negative real numbers a, b, c with $a^2 + b^2 + c^2 = 1$

$$\sqrt{a + b} + \sqrt{a + c} + \sqrt{b + c} \geq 5abc + 2.$$

Day 3 March 3rd

1 Let P be a point inside the acute triangle ABC with $m(\widehat{PAC}) = m(\widehat{PCB})$. D is the midpoint of the segment PC. AP and BC intersect at E, and BP and DE intersect at Q. Prove that $\sin \widehat{BCQ} = \sin \widehat{BAP}$.

2 $a_1 = -5$, $a_2 = -6$ and for all $n \geq 2$ the $(a_n)^\infty{}_{n=1}$ sequence defined as,

$$a_{n+1} = a_n + (a_1 + 1)(2a_2 + 1)(3a_3 + 1)\cdots((n-1)a_{n-1} + 1)((n^2 + n)a_n + 2n + 1)).$$

If a prime p divides $na_n + 1$ for a natural number n, prove that there is a integer m such that $m^2 \equiv 5 (modp)$

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3 At the bottom-left corner of a 2014×2014 chessboard, there are some green worms and at the top-left corner of the same chessboard, there are some brown worms. Green worms can move only to right and up, and brown worms can move only to right and down. After a while, the worms make some moves and all of the unit squares of the chessboard become occupied at least once throughout this process. Find the minimum total number of the worms.
