**Cono Sur Olympiad 1991**

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**Day 1**

**1** Let $A, B$ and $C$ be three non-collinear points and $E$ ($\neq B$) an arbitrary point not in the straight line $AC$. Construct the parallelograms $ABCD$ and $AECF$. Prove that $BE \parallel DF$.

**2** Two people, $A$ and $B$, play the following game: $A$ start choosing a positive integrer number and then, each player in it's turn, say a number due to the following rule:

If the last number said was odd, the player add $7$ to this number;
If the last number said was even, the player divide it by $2$.

The winner is the player that repeats the first number said. Find all numbers that $A$ can choose in order to win. Justify your answer.

**3** It is known that the number of real solutions of the following system if finite. Prove that this system has an even number of solutions:

$$(y^2+6)(x-1)=y(x^2+1)$$

$$(x^2+6)(y-1)=x(y^2+1)$$

**Day 2**

**1** A game consists in $9$ coins (blacks or whites) arrenged in the following position (see picture 1). If you choose $1$ coin on the border of the square, this coin and it's neighbours change their color. If you choose the coin at the centre, it doesn't change it's color, but the other $8$ coins do. Here is an example of $9$ white coins, and the changes of their colors, choosing the coin said: (see picture 2).
Is it possible, starting with $9$ white coins, to have $9$ black coins?.

**2** Given a square $ABCD$ with side $1$, and a square inside $ABCD$ with side $x$, find (in terms of $x$) the radio $r$ of the circle tangent to two sides of $ABCD$ and touches the square with side $x$. (See picture).

**3** Given a positive integrer number $n$ ($n \neq 0$), let $f(n)$ be the average of all the positive divisors of $n$. For example, $f(3)=\frac{1+3}{2}=2$, and $f(12)=\frac{1+2+3+4+6+12}{6}=\frac{14}{3}$.

**a** Prove that $\frac{n+1}{2} \geq f(n) \geq \sqrt{n}$.

**b** Find all $n$ such that $f(n) = \frac{91}{9}$.