Cono Sur Olympiad 2000

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Day 1

1	Call a positive integer *descending* if, reading left to right, each of its digits (other than its leftmost) is less than or equal to the previous digit. For example, 4221 and 751 are descending while 476 and 455 are not descending. Determine whether there exists a positive integer n for which 16^n is descending.
2	The numbers $1, 2, \ldots, 64$ are written in the squares of an 8×8 chessboard, one number to each square. Then 2×2 tiles are placed on the chessboard (without overlapping) so that each tile covers exactly four squares whose numbers sum to less than 100. Find, with proof, the maximum number of tiles that can be placed on the chessboard, and give an example of a distribution of the numbers $1, 2, \ldots, 64$ into the squares of the chessboard that admits this maximum number of tiles.
3	Inside a 2×2 square, lines parallel to a side of the square (both horizontal and vertical) are drawn thereby dividing the square into rectangles. The rectangles are alternately colored black and white like a chessboard. Prove that if the total area of the white rectangles is equal to the total area of the black rectangles, then one can cut out the black rectangles and reassemble them into a 1×2 rectangle.

Day 2

1	In square $ABCD$ (labeled clockwise), let P be any point on BC and construct square $APRS$ (labeled clockwise). Prove that line CR is tangent to the circumcircle of triangle ABC.
2	Consider the following transformation of the Cartesian plane: choose a lattice point and rotate the plane 90° counterclockwise about that lattice point. Is it possible, through a sequence of such transformations, to take the triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$ to the triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$?
3	Is there a positive integer divisible by the product of its digits such that this product is greater than 10^{2000}?