Cono Sur Olympiad 2007

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by Shu

Day 1

1 Find all pairs (x, y) of nonnegative integers that satisfy

$$x^3y + x + y = xy + 2xy^2.$$

2 Given are 100 positive integers whose sum equals their product. Determine the minimum number of 1s that may occur among the 100 numbers.

3 Let ABC be an acute triangle with altitudes AD, BE, CF where D, E, F lie on BC, AC, AB, respectively. Let M be the midpoint of BC. The circumcircle of triangle AEF cuts the line AM at A and X. The line AM cuts the line CF at Y. Let Z be the point of intersection of AD and BX. Show that the lines YZ and BC are parallel.

Day 2

1 Some cells of a 2007×2007 table are colored. The table is *charrua* if none of the rows and none of the columns are completely colored.(a) What is the maximum number k of colored cells that a charrua table can have?
(b) For such k, calculate the number of distinct charrua tables that exist.

2 Let $ABCDE$ be a convex pentagon that satisfies all of the following:
-There is a circle Γ tangent to each of the sides.
-The lengths of the sides are all positive integers.
-At least one of the sides of the pentagon has length 1.
-The side AB has length 2.
Let P be the point of tangency of Γ with AB.
(a) Determine the lengths of the segments AP and BP.
(b) Give an example of a pentagon satisfying the given conditions.

3 Show that for each positive integer n, there is a positive integer k such that the decimal representation of each of the numbers $k, 2k, \ldots, nk$ contains all of the digits $0, 1, 2, \ldots, 9$.