Cono Sur Olympiad 2013

www.artofproblemsolving.com/community/c5483

by Leicich

Day 1

1 Four distinct points are marked in a line. For each point, the sum of the distances from said point to the other three is calculated; getting in total 4 numbers.

Decide whether these 4 numbers can be, in some order:
a) $29, 29, 35, 37$
b) $28, 29, 35, 37$
c) $28, 34, 34, 37$

2 In a triangle ABC, let M be the midpoint of BC and I the incenter of ABC. If IM = IA, find the least possible measure of $\angle AIM$.

3 *Nocycleland* is a country with 500 cities and 2013 two-way roads, each one of them connecting two cities. A city A *neighbors* B if there is one road that connects them, and a city A *quasi-neighbors* B if there is a city C such that A neighbors C and C neighbors B.
It is known that in Nocycleland, there are no pair of cities connected directly with more than one road, and there are no four cities A, B, C and D such that A neighbors B, B neighbors C, C neighbors D, and D neighbors A.
Show that there is at least one city that quasi-neighbors at least 57 other cities.

Day 2

4 Let M be the set of all integers from 1 to 2013. Each subset of M is given one of k available colors, with the only condition that if the union of two different subsets A and B is M, then A and B are given different colors. What is the least possible value of k?

5 Let $d(k)$ be the number of positive divisors of integer k. A number n is called *balanced* if $d(n-1) \leq d(n) \leq d(n+1)$ or $d(n-1) \geq d(n) \geq d(n+1)$.
Show that there are infinitely many balanced numbers.

6 Let $ABCD$ be a convex quadrilateral. Let $n \geq 2$ be a whole number. Prove that there are n triangles with the same area that satisfy all of the following properties:

a) Their interiors are disjoint, that is, the triangles do not overlap.
b) Each triangle lies either in $ABCD$ or inside of it.
c) The sum of the areas of all of these triangles is at least $\frac{4n}{4n+1}$ the area of $ABCD$.