Tuymaada Olympiad 1999

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by orl

Day 1

1 50 knights of King Arthur sat at the Round Table. A glass of white or red wine stood before each of them. It is known that at least one glass of red wine and at least one glass of white wine stood on the table. The king clapped his hands twice. After the first clap every knight with a glass of red wine before him took a glass from his left neighbour. After the second clap every knight with a glass of white wine (and possibly something more) before him gave this glass to the left neughbour of his left neighbour. Prove that some knight was left without wine.

Proposed by A. Khrabrov, incorrect translation from Hungarian

2 Find all polynomials $P(x)$ such that

$$P(x^3+1)=P(x^3)+P(x^2).$$

Proposed by A. Golovanov

3 What maximum number of elements can be selected from the set $\{1,2,3,\dots,100\}$ so that **no** sum of any three selected numbers is equal to a selected number?

Proposed by A. Golovanov

4 Prove the inequality

$$\frac{x}{y^2-z}+\frac{y}{z^2-x}+\frac{z}{x^2-y}>1,$$

where $2<x,y,z<4$.

Proposed by A. Golovanov

Day 2

1 In the triangle ABC we have $\angle ABC=100^\circ$, $\angle ACB=65^\circ$, $M\in AB$, $N\in AC$, and $\angle MCB=55^\circ$, $\angle NBC=80^\circ$. Find $\angle NMC$.

St.Petersburg folklore

2 Can the graphs of a polynomial of degree 20 and the function $y=\frac{1}{x^{40}}$ have exactly 30 points of intersection?

Proposed by K. Kokhas

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3 A sequence of integers $a_0,\ a_1, \dots a_n \dots$ is defined by the following rules: $a_0 = 0,\ a_1 = 1,\ a_{n+1} > a_n$ for each $n \in \mathbb{N}$, and a_{n+1} is the minimum number such that no three numbers among $a_0,\ a_1, \dots a_{n+1}$ form an arithmetical progression. Prove that $a_{2^n} = 3^n$ for each $n \in \mathbb{N}$.

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4 A right parallelepiped (i.e. a parallelepiped one of whose edges is perpendicular to a face) is given. Its vertices have integral coordinates, and no other points with integral coordinates lie on its faces or edges. Prove that the volume of this parallelepiped is a sum of three perfect squares.

Proposed by A. Golovanov
