Tuymaada Olympiad 2000

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by Snakes, mathmanman, orl, silouan, eduard, joao1, parmenides51

– Seniors

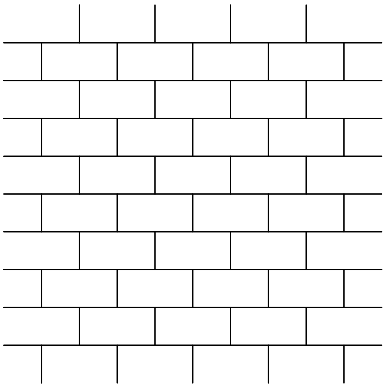
Day 1

1 Let $d(n)$ denote the number of positive divisors of n and let $e(n) = \left[\frac{2000}{n}\right]$ for positive integer n. Prove that

$$d(1) + d(2) + \cdots + d(2000) = e(1) + e(2) + \cdots + e(2000).$$

2 A tangent l to the circle inscribed in a rhombus meets its sides AB and BC at points E and F respectively.
Prove that the product $AE \cdot CF$ is independent of the choice of l.

3 Can the 'brick wall' (infinite in all directions) drawn at the picture be made of wires of length $1, 2, 3, \dots$ (each positive integral length occurs exactly once)? (Wires can be bent but should not overlap; size of a 'brick' is 1×2).



4 Prove for real x_1, x_2, ..., x_n, $0 < x_k \leq \frac{1}{2}$, the inequality

$$\left(\frac{n}{x_1 + \cdots + x_n} - 1\right)^n \leq \left(\frac{1}{x_1} - 1\right) \dots \left(\frac{1}{x_n} - 1\right).$$

Day 2

1 Can the plane be coloured in 2000 colours so that any nondegenerate circle contains points of all 2000 colors?

2 There are 2000 cities in Graphland; some of them are connected by roads.
For every city the number of roads going from it is counted. It is known that there are exactly two equal numbers among all the numbers obtained. What can be these numbers?

3 Polynomial $P(t)$ is such that for all real x,

$$P(\sin x) + P(\cos x) = 1.$$

What can be the degree of this polynomial?

4 Prove that no number of the form 10^{-n}, $n \geq 1,$ can be represented as the sum of reciprocals of factorials of different positive integers.

– Juniors

Day 1

1 Given the number $188188...188$ (number 188 is written 101 times). Some digits of this number are crossed out. What is the largest multiple of 7, that could happen?

2 Is it possible to paint the plane in 4 colors so that inside any circle are the dots of all four colors?

3 Same as Seniors P2

4 Prove that if the product of positive numbers a, b and c equals one, then $\frac{1}{a(a+1)} + \frac{1}{b(b+1)} + \frac{1}{c(c+1)} \geq \frac{3}{2}$

Day 2

5 Are there prime p and q larger than 3, such that $p^2 - 1$ is divisible by q and $q^2 - 1$ divided by p?

6 Let O be the center of the circle circumscribed around the the triangle ABC. The centers of the circles circumscribed around the squares OAB, OBC, OCA lie at the vertices of a regular triangle. Prove that the triangle ABC is right.

7 Every two of five regular pentagons on the plane have a common point.
Is it true that some of these pentagons have a common point?

8 There are 2000 cities in the country, each of which has exactly three roads to other cities. Prove that you can close 1000 roads, so that there is not a single closed route in the country, consisting of an odd number of roads.