Tuymaada Olympiad 2001

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by Snakes, mathmanman, pohoatza, parmenides51

– Seniors

Day 1

1 Ten volleyball teams played a tournament; every two teams met exactly once. The winner of the play gets 1 point, the loser gets 0 (there are no draws in volleyball). If the team that scored n-th has x_n points ($n=1,\dots,10$), prove that $x_1+2x_2+\dots+10x_{10}\geq 165$.

Proposed by D. Teryoshin

2 Solve the equation

$$(a^2,b^2)+(a,bc)+(b,ac)+(c,ab)=199.$$

in positive integers.
(Here (x,y) denotes the greatest common divisor of x and y.)

Proposed by S. Berlov

3 Do there exist quadratic trinomials $P,\ Q,\ R$ such that for every integers x and y an integer z exists satisfying $P(x)+Q(y)=R(z)$?

Proposed by A. Golovanov

4 Unit square $ABCD$ is divided into 10^{12} smaller squares (not necessarily equal). Prove that the sum of perimeters of all the smaller squares having common points with diagonal AC does not exceed 1500.

Proposed by A. Kanel-Belov

Day 2

1 All positive integers are distributed among two disjoint sets N_1 and N_2 such that no difference of two numbers belonging to the same set is a prime greater than 100.

Find all such distributions.

Proposed by N. Sedrakyan

2 Non-zero numbers are arranged in $n\times n$ square ($n>2$). Every number is exactly k times less than the sum of all the other numbers in the same cross (i.e., $2n-2$ numbers written in the

same row or column with this number).
Find all possible k.

Proposed by D. Rostovsky, A. Khrabrov, S. Berlov

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3 $ABCD$ is a convex quadrilateral; half-lines DA and CB meet at point Q; half-lines BA and CD meet at point P. It is known that $\angle AQB = \angle APD$. The bisector of angle $\angle AQB$ meets the sides AB and CD of the quadrilateral at points X and Y, respectively; the bisector of angle $\angle APD$ meets the sides AD and BC at points Z and T, respectively.
The circumcircles of triangles ZQT and XPY meet at point K inside the quadrilateral.
Prove that K lies on the diagonal AC.

Proposed by S. Berlov

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4 Is it possible to colour all positive real numbers by 10 colours so that every two numbers with decimal representations differing in one place only are of different colours? (We suppose that there is no place in a decimal representations such that all digits starting from that place are 9's.)

Proposed by A. Golovanov

– Juniors

Day 1

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1 16 chess players held a tournament among themselves: every two chess players played exactly one game. For victory in the party was given 1 point, for a draw 0.5 points, for defeat 0 points. It turned out that exactly 15 chess players shared the first place. How many points could the sixteenth chess player score?

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2 Is it possible to arrange integers in the cells of the infinite chechered sheet so that every integer appears at least in one cell, and the sum of any 10 numbers in a row vertically or horizontal, would be divisible by 101?

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3 Let ABC be an acute isosceles triangle ($AB = BC$) inscribed in a circle with center O . The line through the midpoint of the chord AB and point O intersects the line AC at L and the circle at the point P. Let the bisector of angle BAC intersects the circle at point K. Lines AB and PK intersect at point D. Prove that the points L, B, D and P lie on the same circle.

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4 Natural numbers $1, 2, 3, .., 100$ are contained in the union of N geometric progressions (not necessarily with integer denominations). Prove that $N \geq 31$

Day 2

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5 Same as Seniors P5

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6 On the side AB of an isosceles triangle AB ($AC = BC$) lie points P and Q such that $\angle PCQ \leq \frac{1}{2}\angle ACB$. Prove that $PQ \leq \frac{1}{2}AB$.

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7 Several rational numbers were written on the blackboard. Dima wrote off their fractional parts on paper. Then all the numbers on the board squared, and Dima wrote off another paper with fractional parts of the resulting numbers. It turned out that on Dima's papers were written the same sets of numbers (maybe in different order). Prove that the original numbers on the board were integers.
(The fractional part of a number x is such a number $\{x\}, 0 \leq \{x\} < 1$, that $x - \{x\}$ is an integer.)

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8 Can three persons, having one double motorcycle, overcome the distance of 70 km in 3 hours? Pedestrian speed is 5 km / h and motorcycle speed is 50 km / h.
