Tuymaada Olympiad 2005

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by perfect_radio, Severius, ciprian, fagot

Day 1

1 The positive integers $1, 2, ..., 121$ are arranged in the squares of a 11×11 table. Dima found the product of numbers in each row and Sasha found the product of the numbers in each column. Could they get the same set of 11 numbers?

Proposed by S. Berlov

2 Six members of the team of Fatalia for the International Mathematical Olympiad are selected from 13 candidates. At the TST the candidates got $a_1, a_2, \ldots, a_{13}$ points with $a_i \neq a_j$ if $i \neq j$.

The team leader has already 6 candidates and now wants to see them and nobody other in the team. With that end in view he constructs a polynomial $P(x)$ and finds the creative potential of each candidate by the formula $c_i = P(a_i)$.

For what minimum n can he always find a polynomial $P(x)$ of degree not exceeding n such that the creative potential of all 6 candidates is strictly more than that of the 7 others?

Proposed by F. Petrov, K. Sukhov

3 The organizers of a mathematical congress found that if they accomodate any participant in a room the rest can be accomodated in double rooms so that 2 persons living in each room know each other. Prove that every participant can organize a round table on graph theory for himself and an even number of other people so that each participant of the round table knows both his neigbours.

Proposed by S. Berlov, S. Ivanov

4 In a triangle ABC, let A_1, B_1, C_1 be the points where the excircles touch the sides BC, CA and AB respectively. Prove that AA_1, BB_1 and CC_1 are the sidelenghts of a triangle.

Proposed by L. Emelyanov

Day 2

5 You have 2 columns of 11 squares in the middle, in the right and in the left you have columns of 9 squares (centered on the ones of 11 squares), then columns of $7, 5, 3, 1$ squares. (This is the way it was explained in the original thread, http://www.artofproblemsolving.com/Forum/viewtopic.php?t= ; anyway, i think you can understand how it looks)

Several rooks stand on the table and beat all the squares (a rook beats the square it stands in, too). Prove that one can remove several rooks such that not more than 11 rooks are left and still beat all the table.

Proposed by D. Rostovsky, based on folklore

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6 Given are a positive integer n and an infinite sequence of proper fractions $x_0 = \frac{a_0}{n}, \ldots, x_i = \frac{a_i}{n+i}$, with $a_i < n + i$. Prove that there exist a positive integer k and integers $c_1, \ldots, c_k$ such that

$$c_1 x_1 + \ldots + c_k x_k = 1.$$

Proposed by M. Dubashinsky

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7 Let I be the incentre of triangle ABC. A circle containing the points B and C meets the segments BI and CI at points P and Q respectively. It is known that $BP \cdot CQ = PI \cdot QI$. Prove that the circumcircle of the triangle PQI is tangent to the circumcircle of ABC.

Proposed by S. Berlov

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8 Let a, b, c be positive reals s.t. $a^2 + b^2 + c^2 = 1$. Prove the following inequality

$$\sum \frac{a}{a^3 + bc} > 3.$$

Proposed by A. Khrabrov
