Tuymaada Olympiad 2006

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by maky, rem

Day 1

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1 Seven different odd primes are given. Is it possible that for any two of them, the difference of their eight powers to be divisible by all the remaining ones ?

Proposed by F. Petrov, K. Sukhov

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2 We call a sequence of integers a *Fibonacci-type sequence* if it is infinite in both ways and $a_n = a_{n-1} + a_{n-2}$ for any $n \in \mathbb{Z}$. How many *Fibonacci-type sequences* can we find, with the property that in these sequences there are two consecutive terms, strictly positive, and less or equal than N ? (two sequences are considered to be the same if they differ only by shifting of indices)

Proposed by I. Pevzner

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3 A line d is given in the plane. Let $B \in d$ and A another point, not on d, and such that AB is not perpendicular on d. Let ω be a variable circle touching d at B and letting A outside, and X and Y the points on ω such that AX and AY are tangent to the circle. Prove that the line XY passes through a fixed point.

Proposed by F. Bakharev

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4 Find all functions $f : (0, \infty) \rightarrow (0, \infty)$ with the following properties: $f(x+1) = f(x) + 1$ and $f\left(\frac{1}{f(x)}\right) = \frac{1}{x}$.

Proposed by P. Volkmann

Day 2

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1 There are 100 boxers, each of them having different strengths, who participate in a tournament. Any of them fights each other only once. Several boxers form a plot. In one of their matches, they hide in their glove a horse shoe. If in a fight, only one of the boxers has a horse shoe hidden, he wins the fight; otherwise, the stronger boxer wins. It is known that there are three boxers who obtained (strictly) more wins than the strongest three boxers. What is the minimum number of plotters ?

Proposed by N. Kalinin

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2 Let ABC be a triangle, G it's centroid, H it's orthocenter, and M the midpoint of the arc $\widehat{AC}$ (not containing B). It is known that $MG = R$, where R is the radius of the circumcircle. Prove that $BG \geq BH$.

Proposed by F. Bakharev

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3 From a $n \times (n-1)$ rectangle divided into unit squares, we cut the *corner*, which consists of the first row and the first column. (that is, the corner has $2n-2$ unit squares). For the following, when we say *corner* we reffer to the above definition, along with rotations and symmetry. Consider an infinite lattice of unit squares. We will color the squares with k colors, such that for any corner, the squares in that corner are coloured differently (that means that there are no squares coloured with the same colour). Find out the minimum of k.

Proposed by S. Berlov

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4 For a positive integer, we define it's *set of exponents* the unordered list of all the exponents of the primes, in it's decomposition. For example, $18 = 2 \cdot 3^2$ has it's set of exponents $1, 2$ and $300 = 2^2 \cdot 3 \cdot 5^2$ has it's set of exponents $1, 2, 2$. There are given two arithmetical progressions $(a_n)_n$ and $(b_n)_n$, such that for any positive integer n, a_n and b_n have the same set of exponents. Prove that the progressions are proportional (that is, there is k such that $a_n = kb_n$ for any n).

Proposed by A. Golovanov
