Tuymaada Olympiad 2010

www.artofproblemsolving.com/community/c5496

by Octav, Shu, joao1

– Grade level 1

Day 1

1 Misha and Sahsa play a game on a 100×100 chessboard. First, Sasha places 50 kings on the board, and Misha places a rook, and then they move in turns, as following (Sasha begins):

At his move, Sasha moves each of the kings one square in any direction, and Misha can move the rook on the horizontal or vertical any number of squares. The kings cannot be captured or stepped over. Sasha's purpose is to capture the rook, and Misha's is to avoid capture.

Is there a winning strategy available for Sasha?

2 Let ABC be an acute triangle, H its orthocentre, D a point on the side $[BC]$, and P a point such that $ADPH$ is a parallelogram.
Show that $\angle BPC > \angle BAC$.

3 Let $f(x) = ax^2+bx+c$ be a quadratic trinomial with a,b,c reals such that any quadratic trinomial obtained by a permutation of f's coefficients has an integer root (including f itself).
Show that $f(1) = 0$.

4 On a blackboard there are 2010 natural nonzero numbers. We define a "move" by erasing x and y with $y \neq 0$ and replacing them with $2x + 1$ and $y - 1$, or we can choose to replace them by $2x + 1$ and $\frac{y-1}{4}$ if $y - 1$ is divisible by 4.

Knowing that in the beginning the numbers 2006 and 2008 have been erased, show that the original set of numbers cannot be attained again by any sequence of moves.

Day 2

1 We have a set M of real numbers with $|M| > 1$ such that for any $x \in M$ we have either $3x - 2 \in M$ or $-4x + 5 \in M$.
Show that M is infinite.

2 We have a number n for which we can find 5 consecutive numbers, none of which is divisible by n, but their product is.
Show that we can find 4 consecutive numbers, none of which is divisible by n, but their product is.

3 Let ABC be a triangle, I its incenter, ω its incircle, P a point such that $PI \perp BC$ and $PA \parallel BC$, $Q \in (AB), R \in (AC)$ such that $QR \parallel BC$ and QR tangent to ω.
Show that $\angle QPB = \angle CPR$.

---

4 Direct one-way flights run between some cities in a certain country. Prove that there is a nonempty set A of cities such that
-there is no flight between any two cities of A; and
-from every city not in A, one can reach some city of A either by a direct flight or by two flights with one change.

– Grade level 2

Day 1

---

1 Misha and Sahsa play a game on a 100×100 chessboard. First, Sasha places 50 kings on the board, and Misha places a rook, and then they move in turns, as following (Sasha begins):

At his move, Sasha moves each of the kings one square in any direction, and Misha can move the rook on the horizontal or vertical any number of squares. The kings cannot be captured or stepped over. Sasha's purpose is to capture the rook, and Misha's is to avoid capture.

Is there a winning strategy available for Sasha?

---

2 In acute triangle ABC, let H denote its orthocenter and let D be a point on side BC. Let P be the point so that $ADPH$ is a parallelogram. Prove that $\angle DCP < \angle BHP$.

---

3 Arranged in a circle are 2010 digits, each of them equal to 1, 2, or 3. For each positive integer k, it's known that in any block of $3k$ consecutive digits, each of the digits appears at most $k+10$ times. Prove that there is a block of several consecutive digits with the same number of 1s, 2s, and 3s.

---

4 Prove that for any positive real number α, the number $\lfloor \alpha n^2 \rfloor$ is even for infinitely many positive integers n.

Day 2

---

1 Baron Mnchausen boasts that he knows a remarkable quadratic triniomial with positive coefficients. The trinomial has an integral root; if all of its coefficients are increased by 1, the resulting trinomial also has an integral root; and if all of its coefficients are also increased by 1, the new trinomial, too, has an integral root. Can this be true?

---

2 For a given positive integer n, it's known that there exist 2010 consecutive positive integers such that none of them is divisible by n but their product is divisible by n. Prove that there exist

2004 consecutive positive integers such that none of them is divisible by n but their product is divisible by n.

---

3 In a cyclic quadrilateral $ABCD$, the extensions of sides AB and CD meet at point P, and the extensions of sides AD and BC meet at point Q. Prove that the distance between the orthocenters of triangles APD and AQB is equal to the distance between the orthocenters of triangles CQD and BPC.

---

4 In a country there are 4^9 schoolchildren living in four cities. At the end of the school year a state examination was held in 9 subjects. It is known that any two students have different marks at least in one subject. However, every two students from the same city got equal marks at least in one subject. Prove that there is a subject such that every two children living in the same city have equal marks in this subject.

Fedor Petrov