Tuymaada Olympiad 2011

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– Grade level 1

Day 1

1 Red, blue, and green children are arranged in a circle. When a teacher asked the red children that have a green neighbor to raise their hands, 20 children raised their hands. When she asked the blue children that have a green neighbor to raise their hands, 25 children raised their hands. Prove that some child that raised her hand had two green neighbors.

2 How many ways are there to remove an 11×11 square from a 2011×2011 square so that the remaining part can be tiled with dominoes (1×2 rectangles)?

3 An excircle of triangle ABC touches the side AB at P and the extensions of sides AC and BC at Q and R, respectively. Prove that if the midpoint of PQ lies on the circumcircle of ABC, then the midpoint of PR also lies on that circumcircle.

4 Prove that, among 100000 consecutive 100-digit positive integers, there is an integer n such that the length of the period of the decimal expansion of $\frac{1}{n}$ is greater than 2011.

Day 2

1 Each real number greater than 1 is coloured red or blue with both colours being used. Prove that there exist real numbers a and b such that the numbers $a+b$ and ab are of different colours.

2 A circle passing through the vertices A and B of a cyclic quadrilateral $ABCD$ intersects diagonals AC and BD at E and F, respectively. The lines AF and BC meet at a point P, and the lines BE and AD meet at a point Q. Prove that PQ is parallel to CD.

3 In a word of more than 10 letters, any two consecutive letters are different. Prove that one can change places of two consecutive letters so that the resulting word is not *periodic*, that is, cannot be divided into equal subwords.

4 The Duke of Squares left to his three sons a square estate, 100×100 square miles, made up of ten thousand 1×1 square mile square plots. The whole estate was divided among his sons as follows. Each son was assigned a point inside the estate. A 1×1 square plot was bequeathed to the son whose assigned point was closest to the center of this square plot. Is it true that, irrespective of the choice of assigned points, each of the regions bequeathed to the sons is connected (that is, there is a path between every two of its points, never leaving the region)?

– Grade level 2

Day 1

1 Red, blue, and green children are arranged in a circle. When a teacher asked the red children that have a green neighbor to raise their hands, 20 children raised their hands. When she asked the blue children that have a green neighbor to raise their hands, 25 children raised their hands. Prove that some child that raised her hand had two green neighbors.

2 Circles ω_1 and ω_2 intersect at points A and B, and M is the midpoint of AB. Points S_1 and S_2 lie on the line AB (but not between A and B). The tangents drawn from S_1 to ω_1 touch it at X_1 and Y_1, and the tangents drawn from S_2 to ω_2 touch it at X_2 and Y_2. Prove that if the line X_1X_2 passes through M, then line Y_1Y_2 also passes through M.

3 Written in each square of an infinite chessboard is the minimum number of moves needed for a knight to reach that square from a given square O. A square is called *singular* if 100 is written in it and 101 is written in all four squares sharing a side with it. How many singular squares are there?

4 In a set of consecutive positive integers, there are exactly 100 perfect cubes and 10 perfect fourth powers. Prove that there are at least 2000 perfect squares in the set.

Day 2

1 Each real number greater than 1 is colored red or blue with both colors being used. Prove that there exist real numbers a and b such that the numbers $a + \frac{1}{b}$ and $b + \frac{1}{a}$ are different colors.

2 In a word of more than 10 letters, any two consecutive letters are different. Prove that one can change places of two consecutive letters so that the resulting word is not *periodic*, that is, cannot be divided into equal subwords.

3 In a convex hexagon $AC'BA'CB'$, every two opposite sides are equal. Let A_1 denote the point of intersection of BC with the perpendicular bisector of AA'. Define B_1 and C_1 similarly. Prove that A_1, B_1, and C_1 are collinear.

4 Let $P(n)$ be a quadratic trinomial with integer coefficients. For each positive integer n, the number $P(n)$ has a proper divisor d_n, i.e., $1 < d_n < P(n)$, such that the sequence $d_1, d_2, d_3, \ldots$ is increasing. Prove that either $P(n)$ is the product of two linear polynomials with integer coefficients or all the values of $P(n)$, for positive integers n, are divisible by the same integer $m > 1$.

