Tuymaada Olympiad 2012

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by mavropnevma, mikolez

– Seniors

Day 1

1 Tanya and Serezha take turns putting chips in empty squares of a chessboard. Tanya starts with a chip in an arbitrary square. At every next move, Serezha must put a chip in the column where Tanya put her last chip, while Tanya must put a chip in the row where Serezha put his last chip. The player who cannot make a move loses. Which of the players has a winning strategy?

Proposed by A. Golovanov

2 Let $P(x)$ be a real quadratic trinomial, so that for all $x \in \mathbb{R}$ the inequality $P(x^3+x) \geq P(x^2+1)$ holds. Find the sum of the roots of $P(x)$.

Proposed by A. Golovanov, M. Ivanov, K. Kokhas

3 Point P is taken in the interior of the triangle ABC, so that

$$\angle PAB = \angle PCB = \frac{1}{4}(\angle A + \angle C).$$

Let L be the foot of the angle bisector of $\angle B$. The line PL meets the circumcircle of $\triangle APC$ at point Q. Prove that QB is the angle bisector of $\angle AQC$.

Proposed by S. Berlov

4 Let $p = 4k+3$ be a prime. Prove that if

$$\frac{1}{0^2+1} + \frac{1}{1^2+1} + \cdots + \frac{1}{(p-1)^2+1} = \frac{m}{n}$$

(where the fraction $\frac{m}{n}$ is in reduced terms), then $p \mid 2m-n$.

Proposed by A. Golovanov

Day 2

1 Solve in positive integers the following equation:

$$\frac{1}{n^2} - \frac{3}{2n^3} = \frac{1}{m^2}$$

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Proposed by A. Golovanov

2 Quadrilateral $ABCD$ is both cyclic and circumscribed. Its incircle touches its sides AB and CD at points X and Y, respectively. The perpendiculars to AB and CD drawn at A and D, respectively, meet at point U; those drawn at X and Y meet at point V, and finally, those drawn at B and C meet at point W. Prove that points U, V and W are collinear.

Proposed by A. Golovanov

3 Prove that for any real numbers a, b, c satisfying $abc = 1$ the following inequality holds

$$\frac{1}{2a^2+b^2+3}+\frac{1}{2b^2+c^2+3}+\frac{1}{2c^2+a^2+3}\le\frac{1}{2}.$$

Proposed by V. Aksenov

4 Integers not divisible by 2012 are arranged on the arcs of an oriented graph. We call the *weight of a vertex* the difference between the sum of the numbers on the arcs coming into it and the sum of the numbers on the arcs going away from it. It is known that the weight of each vertex is divisible by 2012. Prove that non-zero integers with absolute values not exceeding 2012 can be arranged on the arcs of this graph, so that the weight of each vertex is zero.

Proposed by W. Tutte

– Juniors

Day 1

1 Tanya and Serezha take turns putting chips in empty squares of a chessboard. Tanya starts with a chip in an arbitrary square. At every next move, Serezha must put a chip in the column where Tanya put her last chip, while Tanya must put a chip in the row where Serezha put his last chip. The player who cannot make a move loses. Which of the players has a winning strategy?

Proposed by A. Golovanov

2 A rectangle $ABCD$ is given. Segment DK is equal to BD and lies on the half-line DC. M is the midpoint of BK. Prove that AM is the angle bisector of $\angle BAC$.

Proposed by S. Berlov

3 Prove that N^2 arbitrary distinct positive integers ($N > 10$) can be arranged in a $N \times N$ table, so that all $2N$ sums in rows and columns are distinct.

Proposed by S. Volchenkov

4 Let $p = 1601$. Prove that if

$$\frac{1}{0^2+1}+\frac{1}{1^2+1}+\cdots+\frac{1}{(p-1)^2+1}=\frac{m}{n},$$

where we only sum over terms with denominators not divisible by p (and the fraction $\frac{m}{n}$ is in reduced terms) then $p \mid 2m+n$.

Proposed by A. Golovanov

Day 2

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1 The vertices of a regular 2012-gon are labeled $A_1, A_2, \ldots, A_{2012}$ in some order. It is known that if $k+\ell$ and $m+n$ leave the same remainder when divided by 2012, then the chords A_kA_ℓ and A_mA_n have no common points. Vasya walks around the polygon and sees that the first two vertices are labeled A_1 and A_4. How is the tenth vertex labeled?

Proposed by A. Golovanov

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2 Solve in positive integers the following equation:

$$\frac{1}{n^2}-\frac{3}{2n^3}=\frac{1}{m^2}$$

Proposed by A. Golovanov

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3 A circle is contained in a quadrilateral with successive sides of lengths $3, 6, 5$ and 8. Prove that the length of its radius is less than 3.

Proposed by K. Kokhas

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4 25 little donkeys stand in a row; the rightmost of them is Eeyore. Winnie-the-Pooh wants to give a balloon of one of the seven colours of the rainbow to each donkey, so that successive donkeys receive balloons of different colours, and so that at least one balloon of each colour is given to some donkey. Eeyore wants to give to each of the 24 remaining donkeys a pot of one of six colours of the rainbow (except red), so that at least one pot of each colour is given to some donkey (but successive donkeys can receive pots of the same colour). Which of the two friends has more ways to get his plan implemented, and how many times more?

Eeyore is a character in the Winnie-the-Pooh books by A. A. Milne. He is generally depicted as a pessimistic, gloomy, depressed, old grey stuffed donkey, who is a friend of the title character, Winnie-the-Pooh. His name is an onomatopoeic representation of the braying sound made by a normal donkey. Of course, Winnie-the-Pooh is a fictional anthropomorphic bear.

Proposed by F. Petrov

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