Tuymaada Olympiad 2013

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by mavropnevma, micliva

– Seniors

Day 1

1 100 heaps of stones lie on a table. Two players make moves in turn. At each move, a player can remove any non-zero number of stones from the table, so that at least one heap is left untouched. The player that cannot move loses. Determine, for each initial position, which of the players, the first or the second, has a winning strategy.

K. Kokhas

EDIT. It is indeed confirmed by the sender that empty heaps are still heaps, so the third post contains the right guess of an interpretation.

2 Points X and Y inside the rhombus $ABCD$ are such that Y is inside the convex quadrilateral $BXDC$ and $2\angle XBY = 2\angle XDY = \angle ABC$. Prove that the lines AX and CY are parallel.

S. Berlov

3 The vertices of a connected graph cannot be coloured with less than $n+1$ colours (so that adjacent vertices have different colours).

Prove that $\dfrac{n(n-1)}{2}$ edges can be removed from the graph so that it remains connected.

V. Dolnikov

EDIT. It is confirmed by the official solution that the graph is tacitly assumed to be **finite**.

4 Prove that if x, y, z are positive real numbers and $xyz = 1$ then

$$\frac{x^3}{x^2+y} + \frac{y^3}{y^2+z} + \frac{z^3}{z^2+x} \geq \frac{3}{2}.$$

A. Golovanov

Day 2

5 Prove that every polynomial of fourth degree can be represented in the form $P(Q(x))+R(S(x))$, where P, Q, R, S are quadratic trinomials.

A. Golovanov

EDIT. It is confirmed that assuming the coefficients to be **real**, while solving the problem, earned a maximum score.

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6 Solve the equation $p^2 - pq - q^3 = 1$ in prime numbers.

A. Golovanov

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7 Points A_1, A_2, A_3, A_4 are the vertices of a regular tetrahedron of edge length 1. The points B_1 and B_2 lie inside the figure bounded by the plane $A_1A_2A_3$ and the spheres of radius 1 and centres A_1, A_2, A_3.
Prove that $B_1B_2 < \max\{B_1A_1, B_1A_2, B_1A_3, B_1A_4\}$.

A. Kupavsky

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8 Cards numbered from 1 to 2^n are distributed among k children, $1 \le k \le 2^n$, so that each child gets at least one card. Prove that the number of ways to do that is divisible by 2^{k-1} but not by 2^k.

M. Ivanov

– Juniors

Day 1

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1 100 heaps of stones lie on a table. Two players make moves in turn. At each move, a player can remove any non-zero number of stones from the table, so that at least one heap is left untouched. The player that cannot move loses. Determine, for each initial position, which of the players, the first or the second, has a winning strategy.

K. Kokhas

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2 $ABCDEF$ is a convex hexagon, such that in it $AC \parallel DF$, $BD \parallel AE$ and $CE \parallel BF$. Prove that

$$AB^2 + CD^2 + EF^2 = BC^2 + DE^2 + AF^2.$$

N. Sedrakyan

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3 For every positive real numbers a and b prove the inequality

$$\sqrt{ab} \le \frac{1}{3}\sqrt{\frac{a^2+b^2}{2}} + \frac{2}{3}\frac{2}{\frac{1}{a}+\frac{1}{b}}.$$

A. Khabrov

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4 The vertices of a connected graph cannot be coloured with less than $n+1$ colours (so that adjacent vertices have different colours).
Prove that $\dfrac{n(n-1)}{2}$ edges can be removed from the graph so that it remains connected.

V. Dolnikov

EDIT. It is confirmed by the official solution that the graph is tacitly assumed to be **finite**.

Day 2

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5 Each face of a $7 \times 7 \times 7$ cube is divided into unit squares. What is the maximum number of squares that can be chosen so that no two chosen squares have a common point?

A. Chukhnov

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6 Quadratic trinomials with positive leading coefficients are arranged in the squares of a 6×6 table. Their 108 coefficients are all integers from -60 to 47 (each number is used once). Prove that at least in one column the sum of all trinomials has a real root.

K. Kokhas & F. Petrov

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7 Solve the equation $p^2 - pq - q^3 = 1$ in prime numbers.

A. Golovanov

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8 The point A_1 on the perimeter of a convex quadrilateral $ABCD$ is such that the line AA_1 divides the quadrilateral into two parts of equal area. The points B_1, C_1, D_1 are defined similarly.
Prove that the area of the quadrilateral $A_1B_1C_1D_1$ is greater than a quarter of the area of $ABCD$.

L. Emelyanov
