Tuymaada Olympiad 2014

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– Senior League

Day 1

1 Four consecutive three-digit numbers are divided respectively by four consecutive two-digit numbers. What minimum number of different remainders can be obtained?

(A. Golovanov)

2 The points K and L on the side BC of a triangle $\triangle ABC$ are such that $\widehat{BAK} = \widehat{CAL} = 90^\circ$. Prove that the midpoint of the altitude drawn from A, the midpoint of KL and the circumcentre of $\triangle ABC$ are collinear.

(A. Akopyan, S. Boev, P. Kozhevnikov)

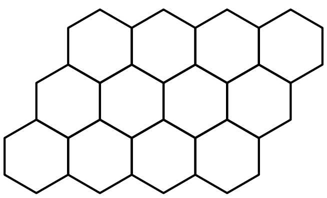
3 Positive numbers $a,\ b,\ c$ satisfy $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=3$. Prove the inequality

$$\frac{1}{\sqrt{a^3+1}}+\frac{1}{\sqrt{b^3+1}}+\frac{1}{\sqrt{c^3+1}}\le\frac{3}{\sqrt{2}}.$$

(N. Alexandrov)

4 A $k\times\ell$ 'parallelogram' is drawn on a paper with hexagonal cells (it consists of k horizontal rows of ℓ cells each). In this parallelogram a set of non-intersecting sides of hexagons is chosen; it divides all the vertices into pairs.

Juniors) How many vertical sides can there be in this set?
Seniors) How many ways are there to do that?



(T. Doslic)

Day 2

5 There is an even number of cards on a table; a positive integer is written on each card. Let a_k be the number of cards having k written on them. It is known that

$$a_n - a_{n-1} + a_{n-2} - \cdots \geq 0$$

for each positive integer n. Prove that the cards can be partitioned into pairs so that the numbers in each pair differ by 1.

(A. Golovanov)

6 Each of n black squares and n white squares can be obtained by a translation from each other. Every two squares of different colours have a common point. Prove that ther is a point belonging at least to n squares.

(V. Dolnikov)

7 A parallelogram $ABCD$ is given. The excircle of triangle $\triangle ABC$ touches the sides AB at L and the extension of BC at K. The line DK meets the diagonal AC at point X; the line BX meets the median CC_1 of trianlge $\triangle ABC$ at Y. Prove that the line YL, median BB_1 of triangle $\triangle ABC$ and its bisector CC' have a common point.

(A. Golovanov)

8 Let positive integers $a,\ b,\ c$ be pairwise coprime. Denote by $g(a,b,c)$ the maximum integer not representable in the form $xa+yb+zc$ with positive integral $x,\ y,\ z$. Prove that

$$g(a,b,c) \geq \sqrt{2abc}$$

(M. Ivanov)

1. It can be proven that $g(a,b,c) \geq \sqrt{3abc}$.
2. The constant 3 is the best possible, as proved by the equation $g(3,3k+1,3k+2) = 9k+5$.

– Junior League

Day 1

1 Given are three different primes. What maximum number of these primes can divide their sum?

(A. Golovanov)

2 A $k \times \ell$ 'parallelogram' is drawn on a paper with hexagonal cells (it consists of k horizontal rows of ℓ cells each). In this parallelogram a set of non-intersecting sides of hexagons is chosen; it divides all the vertices into pairs.

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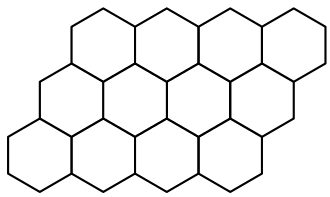

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(A. Akopyan, S. Boev, P. Kozhevnikov)

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(N. Alexandrov)

Day 2

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5 For two quadratic trinomials $P(x)$ and $Q(x)$ there is a linear function $\ell(x)$ such that $P(x) = Q(\ell(x))$ for all real x. How many such linear functions $\ell(x)$ can exist?

(A. Golovanov)

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6 Radius of the circle ω_A with centre at vertex A of a triangle $\triangle ABC$ is equal to the radius of the excircle tangent to BC. The circles ω_B and ω_C are defined similarly. Prove that if two of these circles are tangent then every two of them are tangent to each other.

(L. Emelyanov)

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7 Each of n black squares and n white squares can be obtained by a translation from each other. Every two squares of different colours have a common point. Prove that ther is a point belonging at least to n squares.

(V. Dolnikov)

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8 There are m villages on the left bank of the Lena, n villages on the right bank and one village on an island. It is known that $(m+1,n+1)>1$. Every two villages separated by water are connected by ferry with positive integral number.
The inhabitants of each village say that all the ferries operating in their village have different numbers and these numbers form a segment of the series of the integers. Prove that at least some of them are wrong.

(K. Kokhas)
