Italy TST 1999

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1 Prove that for any prime number p the equation $2^p + 3^p = a^n$ has no solution (a, n) in integers greater than 1.

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2 Let D and E be points on sides AB and AC respectively of a triangle ABC such that DE is parallel to BC and tangent to the incircle of ABC. Prove that

$$DE \le \frac{1}{8}(AB + BC + CA)$$

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3 (a) Find all strictly monotone functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x + f(y)) = f(x) + y \quad \text{for all real } x, y.$$

(b) If $n > 1$ is an integer, prove that there is no strictly monotone function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x + f(y)) = f(x) + y^n \quad \text{for all real } x, y.$$

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4 Let X be an n-element set and let $A_1, \dots, A_m$ be subsets of X such that

i) $|A_i| = 3$ for each $i = 1, \dots, m$.

ii) $|A_i \cap A_j| \le 1$ for any two distinct indices i, j.

Show that there exists a subset of X with at least $\lfloor \sqrt{2n} \rfloor$ elements which does not contain any of the A_is.
