Italy TST 2000

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by WakeUp, outback

1 Determine all triples (x,y,z) of positive integers such that

$$\frac{13}{x^2}+\frac{1996}{y^2}=\frac{z}{1997}$$

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2 Let ABC be an isosceles right triangle and M be the midpoint of its hypotenuse AB. Points D and E are taken on the legs AC and BC respectively such that $AD=2DC$ and $BE=2EC$. Lines AE and DM intersect at F. Show that FC bisects the $\angle DFE$.

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3 Given positive numbers a_1 and b_1, consider the sequences defined by

$$a_{n+1}=a_n+\frac{1}{b_n},\quad b_{n+1}=b_n+\frac{1}{a_n}\quad (n\ge 1)$$

Prove that $a_{25}+b_{25}\geq 10\sqrt{2}$.

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4 On a mathematical competition n problems were given. The final results showed that:
(i) on each problem, exactly three contestants scored 7 points;
(ii) for each pair of problems, exactly one contestant scored 7 points on both problems.
Prove that if $n\ge 8$, then there is a contestant who got 7 points on each problem. Is this statement necessarily true if $n=7$?
