Italy TST 2002

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Day 1

1	Given that in a triangle ABC, $AB = 3$, $BC = 4$ and the midpoints of the altitudes of the triangle are collinear, find all possible values of the length of AC.
2	Prove that for each prime number p and positive integer n, p^n divides $$\binom{p^n}{p} - p^{n-1}.$$
3	Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ which satisfy the following conditions: (i) $f(x + f(y)) = f(x)f(y)$ for all $x, y > 0$; (ii) there are at most finitely many x with $f(x) = 1$.

Day 2

1	A scalene triangle ABC is inscribed in a circle Γ. The bisector of angle A meets BC at E. Let M be the midpoint of the arc BAC. The line ME intersects Γ again at D. Show that the circumcentre of triangle AED coincides with the intersection point of the tangent to Γ at D and the line BC.
2	On a soccer tournament with $n \ge 3$ teams taking part, several matches are played in such a way that among any three teams, some two play a match. (a) If $n = 7$, find the smallest number of matches that must be played. (b) Find the smallest number of matches in terms of n.
3	Prove that for any positive integer m there exist an infinite number of pairs of integers (x, y) such that (i) x and y are relatively prime; (ii) x divides $y^2 + m$; (iii) y divides $x^2 + m$.