Italy TST 2003

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by WakeUp

Day 1

1	Find all triples of positive integers (a, b, p) with a, b positive integers and p a prime number such that $2^a + p^b = 19^a$
2	Let $B \neq A$ be a point on the tangent to circle S_1 through the point A on the circle. A point C outside the circle is chosen so that segment AC intersects the circle in two distinct points. Let S_2 be the circle tangent to AC at C and to S_1 at some point D, where D and B are on the opposite sides of the line AC. Let O be the circumcentre of triangle BCD. Show that O lies on the circumcircle of triangle ABC.
3	Determine all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy $$f(f(x) + y) = 2x + f(f(y) - x) \quad \text{for all real } x, y.$$

Day 2

1	The incircle of a triangle ABC touches the sides AB, BC, CA at points D, E, F respectively. The line through A parallel to DF meets the line through C parallel to EF at G. (a) Prove that the quadrilateral $AICG$ is cyclic. (b) Prove that the points B, I, G are collinear.
2	For n an odd positive integer, the unit squares of an $n \times n$ chessboard are coloured alternately black and white, with the four corners coloured black. A *tromino* is an L-shape formed by three connected unit squares. (a) For which values of n is it possible to cover all the black squares with non-overlapping trominoes lying entirely on the chessboard? (b) When it is possible, find the minimum number of trominoes needed.
3	Let $p(x)$ be a polynomial with integer coefficients and let n be an integer. Suppose that there is a positive integer k for which $f^{(k)}(n) = n$, where $f^{(k)}(x)$ is the polynomial obtained as the composition of k polynomials f. Prove that $p(p(n)) = n$.