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Day 1

1 At the vertices A, B, C, D, E, F, G, H of a cube, $2001, 2002, 2003, 2004, 2005, 2008, 2007$ and 2006 stones respectively are placed. It is allowed to move a stone from a vertex to each of its three neighbours, or to move a stone to a vertex from each of its three neighbours. Which of the following arrangements of stones at $A, B, \ldots, H$ can be obtained? (a) $2001, 2002, 2003, 2004, 2006, 2007, 200$ (b) $2002, 2003, 2004, 2001, 2006, 2005, 2008, 2007$; (c) $2004, 2002, 2003, 2001, 2005, 2008, 2007, 2006.$

2 Let $\mathcal{P}_0 = A_0A_1 \ldots A_{n-1}$ be a convex polygon such that $A_iA_{i+1} = 2^{[i/2]}$ for $i = 0, 1, \ldots, n-1$ (where $A_n = A_0$). Define the sequence of polygons $\mathcal{P}_k = A_0^kA_1^k \ldots A_{n-1}^k$ as follows: A_i^1 is symmetric to A_i with respect to A_0, A_i^2 is symmetric to A_i^1 with respect to A_1^1, A_i^3 is symmetric to A_i^2 with respect to A_2^2 and so on. Find the values of n for which infinitely many polygons $\mathcal{P}_k$ coincide with $\mathcal{P}_0$.

3 Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all $m, n \in \mathbb{N}$,

$$(2^m + 1)f(n)f(2^mn) = 2^mf(n)^2 + f(2^mn)^2 + (2^m - 1)^2n.$$

Day 2

1 Two circles γ_1 and γ_2 intersect at A and B. A line r through B meets γ_1 at C and γ_2 at D so that B is between C and D. Let s be the line parallel to AD which is tangent to γ_1 at E, at the smaller distance from AD. Line EA meets γ_2 in F. Let t be the tangent to γ_2 at F. (a) Prove that t is parallel to AC. (b) Prove that the lines r, s, t are concurrent.

2 A positive integer n is said to be a *perfect power* if $n = a^b$ for some integers a, b with $b > 1$. (a) Find 2004 perfect powers in arithmetic progression. (b) Prove that perfect powers cannot form an infinite arithmetic progression.

3 Given real numbers $x_i, y_i (i = 1, 2, \ldots, n)$, let A be the $n \times n$ matrix given by $a_{ij} = 1$ if $x_i \geq y_j$ and $a_{ij} = 0$ otherwise. Suppose B is a $n \times n$ matrix whose entries are 0 and 1 such that the sum of entries in any row or column of B equals the sum of entries in the corresponding row or column of A. Prove that $B = A$.