Italy TST 2006

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Day 1 May 24th

1 Let S be a string of 99 characters, 66 of which are A and 33 are B. We call S *good* if, for each n such that $1 \leq n \leq 99$, the sub-string made from the first n characters of S has an odd number of distinct permutations. How many good strings are there? Which strings are good?

2 Let ABC be a triangle, let H be the orthocentre and L, M, N the midpoints of the sides AB, BC, CA respectively. Prove that

$$HL^2 + HM^2 + HN^2 < AL^2 + BM^2 + CN^2$$

if and only if ABC is acute-angled.

3 Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that for all integers m, n,

$$f(m - n + f(n)) = f(m) + f(n).$$

Day 2

1 The circles γ_1 and γ_2 intersect at the points Q and R and internally touch a circle γ at A_1 and A_2 respectively. Let P be an arbitrary point on γ. Segments PA_1 and PA_2 meet γ_1 and γ_2 again at B_1 and B_2 respectively.
a) Prove that the tangent to γ_1 at B_1 and the tangent to γ_2 at B_2 are parallel.
b) Prove that B_1B_2 is the common tangent to γ_1 and γ_2 iff P lies on QR.

2 Let n be a positive integer, and let A_n be the the set of all positive integers $a \leq n$ such that $n | a^n + 1$.
a) Find all n such that $A_n \neq \emptyset$
b) Find all n such that $|A_n|$ is even and non-zero.
c) Is there n such that $|A_n| = 130$?

3 Let $P(x)$ be a polynomial with complex coefficients such that $P(0) \neq 0$. Prove that there exists a multiple of $P(x)$ with real positive coefficients if and only if $P(x)$ has no real positive root.