Italy TST 2007

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Day 1

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1 Let ABC an acute triangle.

(a) Find the locus of points that are centers of rectangles whose vertices lie on the sides of ABC;

(b) Determine if exist some points that are centers of 3 distinct rectangles whose vertices lie on the sides of ABC.

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2 In a competition, there were $2n+1$ teams. Every team plays exatly once against every other team. Every match finishes with the victory of one of the teams. We call cyclical a 3-subset of team A, B, C if A won against B, B won against C , C won against A.

(a) Find the minimum of cyclical 3-subset (depending on n);
(b) Find the maximum of cyclical 3-subset (depending on n).

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3 Find all $f : R \longrightarrow R$ such that

$$f(xy + f(x)) = xf(y) + f(x)$$

for every pair of real numbers x, y.

Day 2

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1 We have a complete graph with n vertices. We have to color the vertices and the edges in a way such that: no two edges pointing to the same vertice are of the same color; a vertice and an edge pointing him are coloured in a different way. What is the minimum number of colors we need?

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2 Let ABC a acute triangle.

(a) Find the locus of all the points P such that, calling O_a, O_b, O_c the circumcenters of PBC, PAC, PAB:

$$\frac{O_aO_b}{AB} = \frac{O_bO_c}{BC} = \frac{O_cO_a}{CA}$$

(b) For all points P of the locus in (a), show that the lines AO_a, BO_b , CO_c are cuncurrent (in X);

(c) Show that the power of X wrt the circumcircle of ABC is:

$$-\frac{a^2+b^2+c^2-5R^2}{4}$$

Where $a=BC$, $b=AC$ and $c=AB$.

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3 Let $p\geq 5$ be a prime.

(a) Show that exists a prime $q\neq p$ such that $q|(p-1)^p+1$

(b) Factoring in prime numbers $(p-1)^p+1=\prod_{i=1}^n p_i^{a_i}$ show that:

$$\sum_{i=1}^n p_i a_i \geq \frac{p^2}{2}$$
