Italy TST 2009

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Day 1

1 Let n, k be positive integers such that $n \geq k$. n lamps are placed on a circle, which are all off. In any step we can change the state of k consecutive lamps. In the following three cases, how many states of lamps are there in all 2^n possible states that can be obtained from the initial state by a certain series of operations?
i)k is a prime number greater than 2;
ii) k is odd;
iii) k is even.

2 ABC is a triangle in the plane. Find the locus of point P for which PA, PB, PC form a triangle whose area is equal to one third of the area of triangle ABC.

3 Find all pairs of integers (x, y) such that

$$y^3 = 8x^6 + 2x^3y - y^2.$$

Day 2

1 Let n be an even positive integer. An n-degree monic polynomial $P(x)$ has n real roots (not necessarily distinct). Suppose y is a positive real number such that for any real number $t < y$, we have $P(t) > 0$. Prove that

$$P(0)^{\frac{1}{n}} - P(y)^{\frac{1}{n}} \geq y.$$

2 Two circles O_1 and O_2 intersect at M, N. The common tangent line nearer to M of the two circles touches O_1, O_2 at A, B respectively. Let C, D be the symmetric points of A, B with respect to M respectively. The circumcircle of triangle DCM intersects circles O_1 and O_2 at points E, F respectively which are distinct from M. Prove that the circumradii of the triangles MEF and NEF are equal.

3 Two persons, A and B, set up an incantation contest in which they spell incantations (i.e. a finite sequence of letters) alternately. They must obey the following rules:
i) Any incantation can appear no more than once;
ii) Except for the first incantation, any incantation must be obtained by permuting the letters

of the last one before it, or deleting one letter from the last incantation before it;
iii)The first person who cannot spell an incantation loses the contest. Answer the following questions:
a) If A says '$STAGEPREIMO$' first, then who will win?
b) Let M be the set of all possible incantations whose lengths (i.e. the numbers of letters in them) are 2009 and containing only four letters A, B, C, D, each of them appearing at least once. Find the first incantation (arranged in dictionary order) in M such that A has a winning strategy by starting with it.
