**Nordic 1998**

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**1** Determine all functions $f$ defined in the set of rational numbers and taking their values in the same set such that the equation $f(x+y)+f(x-y)=2f(x)+2f(y)$ holds for all rational numbers $x$ and $y$.

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**2** Let $C_1$ and $C_2$ be two circles intersecting at $A$ and $B$. Let $S$ and $T$ be the centres of $C_1$ and $C_2$, respectively. Let $P$ be a point on the segment $AB$ such that $|AP| \neq |BP|$ and $P \neq A, P \neq B$. We draw a line perpendicular to $SP$ through $P$ and denote by $C$ and $D$ the points at which this line intersects $C_1$. We likewise draw a line perpendicular to $TP$ through $P$ and denote by $E$ and F the points at which this line intersects $C_2$. Show that $C, D, E,$ and $F$ are the vertices of a rectangle.

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**3** (a) For which positive numbers $n$ does there exist a sequence $x_1, x_2, ..., x_n$, which contains each of the numbers $1, 2, ..., n$ exactly once and for which $x_1 + x_2 + ... + x_k$ is divisible by $k$ for each $k = 1, 2, ...., n$?
(b) Does there exist an infinite sequence $x_1, x_2, x_3, ...$, which contains every positive integer exactly once and such that $x_1 + x_2 + ... + x_k$ is divisible by $k$ for every positive integer $k$?

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**4** Let $n$ be a positive integer. Count the number of numbers $k \in \{0, 1, 2, ..., n\}$ such that $\binom{n}{k}$ is odd. Show that this number is a power of two, i.e. of the form $2^p$ for some nonnegative integer $p$.

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