

Saint Petersburg Mathematical Olympiad 2014

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by RagvaloD

– **Grade 11**

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- 1** Let $f(x)$ is such function, that $f(x) = 1$ for integer x and $f(x) = 0$ for non integer x . Build such function using only variable x , integer numbers, and operations $+$, $-$, $*$, $/$, $[\cdot]$ (plus, minus, multiply, divide and integer part)
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- 2** There are cities in country, and some cities are connected by roads. Not more than 100 roads go from every city. Set of roads is called as ideal if all roads in set have not common ends, and we can not add one more road in set without breaking this rule. Every day minister destroy one ideal set of roads. Prove, that he need not more than 199 days to destroy all roads in country.
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- 3** N in natural. There are natural numbers from N^3 to $N^3 + N$ on the board. a numbers was colored in red, b numbers was colored in blue. Sum of red numbers in divisible by sum of blue numbers. Prove, that $b|a$
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- 4** Points B_1, C_1 are on AC and AB and $B_1C_1 \parallel BC$. Circumcircle of ABB_1 intersect CC_1 at L . Circumcircle CLB_1 is tangent to AL .
Prove $AL \leq \frac{AC+AC_1}{2}$
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- 5** M is infinite set of natural numbers. If $a, b, a \neq b$ are in M , then $a^b + 2$ or $a^b - 2$ (or both) are in M . Prove that there is composite number in M
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- 6** In the $n \times n$ table in every cell there is one child. Every child looks in neighbour cell. So every child sees ear or back of the head of neighbour. What is minimal number children, that see ear ?
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- 7** I - incenter, M - midpoint of arc BAC of circumcircle, AL - angle bisector of triangle ABC . MI intersect circumcircle in K . Circumcircle of AKL intersect BC at L and P .
Prove that $\angle AIP = 90$

– **Grade 10**

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- 1** $f(x)$ is square polynomial and $a \neq b$ such that $f(a) = b, f(b) = a$. Prove that there is not other pair (c, d) that $f(c) = d, f(d) = c$
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- 2** There are 40 points on the two parallel lines. We divide it to pairs, such that line segments, that connects point in pair, do not intersect each other (endpoint from one segment cannot lies on

another segment). Prove, that number of ways to do it is less than 3^{39}

3 D is inner point of triangle ABC . E is on BD and $CE = BD$. $\angle ABD = \angle ECD = 10$, $\angle BAD = 40$, $\angle CED = 60$ Prove, that $AB > AC$

4 $a_1 \geq a_2 \geq \dots \geq a_{100n} > 0$
If we take from $(a_1, a_2, \dots, a_{100n})$ some $2n + 1$ numbers $b_1 \geq b_2 \geq \dots \geq b_{2n+1}$ then $b_1 + \dots + b_n > b_{n+1} + \dots + b_{2n+1}$ Prove, that

$$(n + 1)(a_1 + \dots + a_n) > a_{n+1} + a_{n+2} + \dots + a_{100n}$$

5 Incircle ω of ABC touch AC at B_1 . Point E, F on the ω such that $\angle AEB_1 = \angle B_1FC = 90$. Tangents to ω at E, F intersects in D , and B and D are on different sides for line AC . M - midpoint of AC .
Prove, that AE, CF, DM intersects at one point.

6 Same as Grade 11 P5

7 Some cities in country are connected with oneway road. It is known that every closed cyclic route, that don't break traffic laws, consists of even roads. Prove that king of city can place military bases in some cities such that there are not roads between these cities, but for every city without base we can go from city with base by no more than 1 road.

I think it should be one more condition, like there is cycle that connect all cities

– **Grade 9**

1 Same as Grade 10 P1

2 All angles of ABC are in $(30, 90)$. Circumcenter of ABC is O and circumradius is R . Point K is projection of O to angle bisector of $\angle B$, point M is midpoint AC . It is known, that $2KM = R$. Find $\angle B$

3 100 deputies formed 450 commissions. Each two commissions has no more than three common deputies, and every 5 - no more than one. Prove that, that there are 4 commissions that has exactly one common deputy each.

4 We call a natural number venerable if the sum of all its divisors, including 1, but not including the number itself, is 1 less than this number. Find all the venerable numbers, some exact degree of which is also venerable.

5 On a cellular plane with a cell side equal to 1, arbitrarily 100×100 napkin is thrown. It covers some nodes (the node lying on the border of a napkin, is also considered covered). What is

the smallest number of lines (going not necessarily along grid lines) you can certainly cover all these nodes?

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- 6 Points A, B are on circle ω . Points C and D are moved on the arc AB , such that CD has constant length. I_1, I_2 - incenters of ABC and ABD .
Prove that line I_1I_2 is tangent to some fixed circle.
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- 7 Natural a, b, c are pairwise prime. There is infinite table with one integer number in every cell. Sum of numbers in every $a \times a$, every $b \times b$, every $c \times c$ squares is even.
Is it true, that every number in table must be even?
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