Saint Petersburg Mathematical Olympiad 2015

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– **Grade 11**

1 x, y are real numbers such that

$$x^2+y^2=1, 20x^3-15x=3$$

Find the value of $|20y^3-15y|$.(K. Tyshchuk)

2 $a, b>1$ - are naturals, and $a^2+b, a+b^2$ are primes. Prove $(ab+1, a+b)=1$

3 There are weights with mass $1, 3, 5, ..., 2i+1, ...$ Let $A(n)$ -is number of different sets with total mass equal n(For example $A(9)=2$, because we have two sets $9=9=1+3+5$). Prove that $A(n) \leq A(n+1)$ for $n>1$

4 $ABCD$ is convex quadrilateral. Circumcircle of ABC intersect AD and DC at points P and Q. Circumcircle of ADC intersect AB and BC at points S and R. Prove that if $PQRS$ is parallelogram then $ABCD$ is parallelogram

5 Square with side 100 was cut by 99 horizontal and 99 vertical lines into 10000 rectangles (not necessarily with integer sides). How many rectangles in this square with area not exceeding 1 at least can be?

6 In country there are some cities, some pairs of cities are connected with roads.From every city go out exactly 100 roads. We call 10 roads, that go out from same city, as bunch. Prove, that we can split all roads in several bunches.

7 Let BL be angle bisector of acute triangle ABC.Point K choosen on BL such that $\measuredangle AKC - \measuredangle ABC = 90$.point S lies on the extention of BL from L such that $\measuredangle ASC = 90$.Point T is diametrically opposite the point K on the circumcircle of $\triangle AKC$.Prove that ST passes through midpoint of arc ABC.(S. Berlov)
:trampoline: my 100th post :trampoline:

– **Grade 10**

1 There is child camp with some rooms. Call room as $4-$room, if 4 children live here. Not less then half of all rooms are $4-$rooms , other rooms are $3-$rooms. Not less than $2/3$ girls live in $3-$rooms. Prove that not less than 35% of all children are boys.

2 The beaver is chess piece that move to 2 cells by horizontal or vertical. Every cell of 100×100 chessboard colored in some color,such that we can not get from one cell to another with same color with one move of beaver or knight. What minimal color do we need?

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3 $ABCD$ - convex quadrilateral. Bisectors of angles A and D intersect in K, Bisectors of angles B and C intersect in L. Prove

$$2KL \geq |AB - BC + CD - DA|$$

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4 A positive integer n is called *Olympic*, if there exists a quadratic trinomial with integer coeffecients $f(x)$ satisfying $f(f(\sqrt{n})) = 0$. Determine, with proof, the largest Olympic number not exceeding 2015.

A. Khrabrov

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5 $ABCDE$ is convex pentagon. $\angle BCA = \angle BEA = \frac{\angle BDA}{2}, \angle BDC = \angle EDA$.
Prove, that $\angle DEB = \angle DAC$

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6 A sequence of integers is defined as follows: $a_1 = 1, a_2 = 2, a_3 = 3$ and for $n > 3$,

$a_n =$ The smallest integer not occurring earlier, which is relatively prime to a_{n-1} but not relatively prime t

Prove that every natural number occurs exactly once in this sequence.

M. Ivanov

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7 There is convex $n-$gon. We color all its sides and also diagonals, that goes out from one vertex. So we have $2n - 3$ colored segments. We write positive numbers on colored segments. In one move we can take quadrilateral $ABCD$ such, that AC and all sides are colored, then remove AC and color BD with number $\frac{xz+yt}{w}$, where x, y, z, t, w - numbers on AB, BC, CD, DA, AC. After some moves we found that all colored segments are same that was at beginning. Prove, that they have same number that was at beginning.

– **Grade 9**

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1 Is there a quadratic trinomial $f(x)$ with integer coefficients such that $f(f(\sqrt{2})) = 0$?

A. Khrabrov

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2 $AB = CD, AD \parallel BC$ and $AD > BC$. Ω is circumcircle of $ABCD$. Point E is on Ω such that $BE \perp AD$. Prove that $AE + BC > DE$

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3 All cells of 2015×2015 table colored in one of 4 colors. We count number of ways to place one tetris T-block in table. Prove that T-block has cell of all 4 colors in less than 51% of total number of ways.

4 Positive numbers x, y, z satisfy the condition

$$xy + yz + zx + 2xyz = 1.$$

Prove that $4x + y + z \geq 2$.

A. Khrabrov

5 Same as Grade 11 P4

6 There are 10^{2015} planets in an Intergalactic empire. Every two planets are connected by a two-way space line served by one of 2015 travel companies. The Emperor would like to close k of these companies such that it is still possible to reach any planet from any other planet. Find the maximum value of k for which this is always possible.

(D. Karpov)

7 Same as Grade 10 P6