Greece National Olympiad 2015

www.artofproblemsolving.com/community/c55281

by gavrilos

1 Find all triplets (x, y, p) of positive integers such that p be a prime number and $\frac{xy^3}{x+y} = p$

---

2 Let $P(x) = ax^3 + (b-a)x^2 - (c+b)x + c$ and $Q(x) = x^4 + (b-1)x^3 + (a-b)x^2 - (c+a)x + c$ be polynomials of x with a, b, c non-zero real numbers and $b > 0$.If $P(x)$ has three distinct real roots x_0, x_1, x_2 which are also roots of $Q(x)$ then:
A)Prove that $abc > 28$,
B)If a, b, c are non-zero integers with $b > 0$,find all their possible values.

---

3 Given is a triangle ABC with $\angle B = 105^\circ$.Let D be a point on BC such that $\angle BDA = 45^\circ$.

A) If D is the midpoint of BC then prove that $\angle C = 30^\circ$,
B) If $\angle C = 30^\circ$ then prove that D is the midpoint of BC

---

4 Square $ABCD$ with side-length n is divided into n^2 small (fundamental) squares by drawing lines parallel to its sides (the case $n = 5$ is presented on the diagram).The squares' vertices that lie inside (or on the boundary) of the triangle ABD are connected with each other with arcs.Starting from A,we move only upwards or to the right.Each movement takes place on the segments that are defined by the fundamental squares and the arcs of the circles.How many possible roots are there in order to reach C;
