Korean Mathematical Olympiad held in 2017

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– Day 1

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problem 1 Denote U as the set of 20 diagonals of the regular polygon $P_1P_2P_3P_4P_5P_6P_7P_8$. Find the number of sets S which satisfies the following conditions.

1. S is a subset of U.

2. If $P_iP_j \in S$ and $P_jP_k \in S$, and $i \neq k$, $P_iP_k \in S$.

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problem 2 Find all primes p such that there exist an integer n and positive integers k, m which satisfies the following.

$$\frac{(mk^2+2)p-(m^2+2k^2)}{mp+2} = n^2$$

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problem 3 Let there be a scalene triangle ABC, and its incircle hits BC, CA, AB at D, E, F. The perpendicular bisector of BC meets the circumcircle of ABC at P, Q, where P is on the same side with A with respect to BC. Let the line parallel to AQ and passing through D meet EF at R. Prove that the intersection between EF and PQ lies on the circumcircle of BCR.

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problem 4 Let $f : \mathbb{R} \to \mathbb{R}$ be the function as

$$f(x) = \begin{cases} \frac{1}{x-1} & (x > 1) \\ 1 & (x = 1) \\ \frac{x}{1-x} & (x < 1) \end{cases}$$

Let x_1 be a positive irrational number which is a zero of a quadratic polynomial with integer coefficients. For every positive integer n, let $x_{n+1} = f(x_n)$. Prove that there exists different positive integers k and ℓ such that $x_k = x_\ell$.

– Day 2

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problem 5 Given a prime p, show that there exist two integers a, b which satisfies the following.

For all integers m, $m^3 + 2017am + b$ is not a multiple of p.

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problem 6 In a quadrilateral $ABCD$, we have $\angle ACB = \angle ADB = 90$ and $CD < BC$. Denote E as the intersection of AC and BD, and let the perpendicular bisector of BD hit BC at F. The circle with center F which passes through B hits AB at $P(\neq B)$ and AC at Q. Let M be the midpoint of EP. Prove that the circumcircle of EPQ is tangent to AB if and only if B, M, Q are colinear.

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problem 7 Find all real numbers c such that there exists a function $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ which satisfies the following.

For all nonnegative reals x, y, $f(x + y^2) \geq cf(x) + y$.

Here $\mathbb{R}_{\geq 0}$ is the set of all nonnegative reals.

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problem 8 For a positive integer n, there is a school with $2n$ people. For a set X of students in this school, if any two students in X know each other, we call X *well-formed*. If the maximum number of students in a well-formed set is no more than n, find the maximum number of well-formed set.

Here, an empty set and a set with one student is regarded as well-formed as well.
