2018 China National Olympiad

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Day 1 November 15, 2017

1 Let n be a positive integer. Let A_n denote the set of primes p such that there exists positive integers a, b satisfying

$$\frac{a+b}{p} \text{ and } \frac{a^n+b^n}{p^2}$$

are both integers that are relatively prime to p. If A_n is finite, let $f(n)$ denote $|A_n|$.

a) Prove that A_n is finite if and only if $n \neq 2$.

b) Let m, k be odd positive integers and let d be their gcd. Show that

$$f(d) \leq f(k) + f(m) - f(km) \leq 2f(d).$$

2 Let n and k be positive integers and let

$$T = \{(x, y, z) \in \mathbb{N}^3 \mid 1 \leq x, y, z \leq n\}$$

be the length n lattice cube. Suppose that $3n^2 - 3n + 1 + k$ points of T are colored red such that if P and Q are red points and PQ is parallel to one of the coordinate axes, then the whole line segment PQ consists of only red points.

Prove that there exists at least k unit cubes of length 1, all of whose vertices are colored red.

3 Let q be a positive integer which is not a perfect cube. Prove that there exists a positive constant C such that for all natural numbers n, one has

$$\{nq^{\frac{1}{3}}\} + \{nq^{\frac{2}{3}}\} \geq Cn^{-\frac{1}{2}}$$

where $\{x\}$ denotes the fractional part of x.

Day 2 November 16, 2017

4 $ABCD$ is a cyclic quadrilateral whose diagonals intersect at P. The circumcircle of $\triangle APD$ meets segment AB at points A and E. The circumcircle of $\triangle BPC$ meets segment AB at points B and F. Let I and J be the incenters of $\triangle ADE$ and $\triangle BCF$, respectively. Segments IJ and AC meet at K. Prove that the points A, I, K, E are cyclic.

5 Let $n \geq 3$ be an odd number and suppose that each square in a $n \times n$ chessboard is colored either black or white. Two squares are considered adjacent if they are of the same color and share a common vertex and two squares a, b are considered connected if there exists a sequence of squares $c_1, \ldots, c_k$ with $c_1 = a, c_k = b$ such that c_i, c_{i+1} are adjacent for $i = 1, 2, \ldots, k-1$.

Find the maximal number M such that there exists a coloring admitting M pairwise disconnected squares.

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6 Let $n > k$ be two natural numbers and let $a_1, \ldots, a_n$ be real numbers in the open interval $(k-1, k)$. Let $x_1, \ldots, x_n$ be positive reals such that for any subset $I \subset \{1, \ldots, n\}$ satisfying $|I| = k$, one has

$$\sum_{i \in I} x_i \leq \sum_{i \in I} a_i.$$

Find the largest possible value of $x_1 x_2 \cdots x_n$.
