Team Selection Test

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Day 1 March 28th 2015

1 Let l, m, n be positive integers and p be prime. If $p^{2l-1}m(mn+1)^2 + m^2$ is a perfect square, prove that m is also a perfect square.

2 There are 2015 points on a plane and no two distances between them are equal. We call the closest 22 points to a point its $neighbours$. If k points share the same neighbour, what is the maximum value of k?

3 Let m, n be positive integers. Let $S(n, m)$ be the number of sequences of length n and consisting of 0 and 1 in which there exists a 0 in any consecutive m digits. Prove that

$$S(2015n, n).S(2015m, m) \geq S(2015n, m).S(2015m, n)$$

Day 2 March 29th 2015

4 Let ABC be a triangle such that $|AB| = |AC|$ and let D, E be points on the minor arcs $\overset{\frown}{AB}$ and $\overset{\frown}{AC}$ respectively. The lines AD and BC intersect at F and the line AE intersects the circumcircle of $\triangle FDE$ a second time at G. Prove that the line AC is tangent to the circumcircle of $\triangle ECG$.

5 We are going to colour the cells of a 2015×2015 board such that there are none of the following: $1)$ Three cells with the same colour where two of them are in the same column, and the third is in the same row and to the right of the upper cell, $2)$ Three cells with the same colour where two of them are in the same column, and the third is in the same row and to the left of the lower cell.
What is the minimum number of colours k required to make such a colouring possible?

6 Prove that there are infinitely many positive integers n such that $(n!)^{n+2015}$ divides $(n^2)!$.

Day 3 March 30th 2015

7 Find all the functions $f : R \to R$ such that

$$f(x^2) + 4y^2 f(y) = (f(x-y) + y^2)(f(x+y) + f(y))$$

for every real x, y.

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8 Let ABC be a triangle with incenter I and circumcenter O such that $|AC| > |BC| > |AB|$ and the incircle touches the sides BC, CA, AB at D, E, F respectively. Let the reflection of A with respect to F and E be F_1 and E_1 respectively. The circle tangent to BC at D and passing through F_1 intersects AB a second time at F_2 and the circle tangent to BC at D and passing through E_1 intersects AC a second time at E_2. The midpoints of the segments $|OE|$ and $|IF|$ are P and Q respectively. Prove that

$$|AB| + |AC| = 2|BC| \iff PQ \perp E_2F_2$$

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9 In a country with 2015 cities there is exactly one two-way flight between each city. The three flights made between three cities belong to at most two different airline companies. No matter how the flights are shared between some number of companies, if there is always a city in which k flights belong to the same airline, what is the maximum value of k?
