Mexico National Olympiad 2017

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by juckter

1 A knight is placed on each square of the first column of a 2017×2017 board. A *move* consists in choosing two different knights and moving each of them to a square which is one knight-step away. Find all integers k with $1 \leq k \leq 2017$ such that it is possible for each square in the k-th column to contain one knight after a finite number of moves.

Note: Two squares are a knight-step away if they are opposite corners of a 2×3 or 3×2 board.

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2 A set of n positive integers is said to be *balanced* if for each integer k with $1 \leq k \leq n$, the average of any k numbers in the set is an integer. Find the maximum possible sum of the elements of a balanced set, all of whose elements are less than or equal to 2017.

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3 Let ABC be an acute triangle with orthocenter H. The circle through B, H, and C intersects lines AB and AC at D and E respectively, and segment DE intersects HB and HC at P and Q respectively. Two points X and Y, both different from A, are located on lines AP and AQ respectively such that X, H, A, B are concyclic and Y, H, A, C are concyclic. Show that lines XY and BC are parallel.

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4 A subset B of $\{1, 2, \dots, 2017\}$ is said to have property T if any three elements of B are the sides of a nondegenerate triangle. Find the maximum number of elements that a set with property T may contain.

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5 On a circle Γ, points A, B, N, C, D, M are chosen in a clockwise order in such a way that N and M are the midpoints of clockwise arcs BC and AD respectively. Let P be the intersection of AC and BD, and let Q be a point on line MB such that PQ is perpendicular to MN. Point R is chosen on segment MC such that $QB = RC$, prove that the midpoint of QR lies on AC.

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6 Let $n \geq 2$ and m be positive integers. m ballot boxes are placed in a line. Two players A and B play by turns, beginning with A, in the following manner. Each turn, A chooses two boxes and places a ballot in each of them. Afterwards, B chooses one of the boxes, and removes every ballot from it. A wins if after some turn of B, there exists a box containing n ballots. For each n, find the minimum value of m such that A can guarantee a win independently of how B plays.
