Czech And Slovak Mathematical Olympiad, Round III, Category A 2015

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by Radar

1 Find all 4-digit numbers n, such that $n = pqr$, where $p < q < r$ are distinct primes, such that $p + q = r - q$ and $p + q + r = s^2$, where s is a prime number.

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2 Let $A = [0, 0]$ and $B = [n, n]$. In how many ways can we go from A to B, if we always want to go from lattice point to its neighbour (i.e. point with one coordinate the same and one smaller or bigger by one), we never want to visit the same point twice and we want our path to have length $2n + 2$?
(For example, path $[0, 0], [0, 1], [-1, 1], [-1, 2], [0, 2], [1, 2], [2, 2], [2, 3], [3, 3]$ is one of the paths for $n = 3$)

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3 In triangle $\triangle ABC$ with median from B not perpendicular to AB nor BC, we call X and Y points on AB and BC, which lie on the axis of the median from B. Find all such triangles, for which A, C, X, Y lie on one circumrefference.

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4 Find all real triples (a, b, c), for which

$$a(b^2 + c) = c(c + ab)$$
$$b(c^2 + a) = a(a + bc)$$
$$c(a^2 + b) = b(b + ca).$$

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5 In given triangle $\triangle ABC$, difference between sizes of each pair of sides is at least $d > 0$. Let G and I be the centroid and incenter of $\triangle ABC$ and r be its inradius. Show that

$$[AIG] + [BIG] + [CIG] \geq \frac{2}{3}dr,$$

where $[XYZ]$ is (nonnegative) area of triangle $\triangle XYZ$.

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6 Integer $n > 2$ is given. Find the biggest integer d, for which holds, that from any set S consisting of n integers, we can find three different (but not necesarilly disjoint) nonempty subsets, such that sum of elements of each of them is divisible by d.
