Math Prize for Girls Problems 2017

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by Ravi B

1 A bag contains 4 red marbles, 5 yellow marbles, and 6 blue marbles. Three marbles are to be picked out randomly (without replacement). What is the probability that exactly two of them have the same color?

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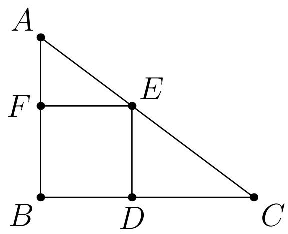
2 In the figure below, $BDEF$ is a square inscribed in $\triangle ABC$. If $\frac{AB}{BC} = \frac{4}{5}$, what is the area of $BDEF$ divided by the area of $\triangle ABC$?



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3 If A and B are numbers such that the polynomial $x^{2017} + Ax + B$ is divisible by $(x+1)^2$, what is the value of B?

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4 If $\mathrm{MATH} + \mathrm{WITH} = \mathrm{GIRLS}$, compute the smallest possible value of GIRLS. Here MATH and WITH are 4-digit numbers and GIRLS is a 5-digit number (all with nonzero leading digits). Different letters represent different digits.

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5 The New York Public Library requires patrons to choose a 4-digit Personal Identification Number (PIN) to access its online system. (Leading zeros are allowed.) The PIN is not allowed to contain either of the following two forbidden patterns:
* A digit that is repeated 3 or more times in a row. For example, 0001 and 5555 are not PINs, but 0010 is a PIN.
* A pair of digits that is duplicated. For example, 1212 and 6363 are not PINs, but 1221 and 6633 are PINs.
How many distinct possible PINs are there?

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6 Let b and c be integers chosen randomly (uniformly and independently) from the set

$$\{-6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6\}.$$

(Note that b and c can be equal.) What is the probability that the two roots of the quadratic $x^2 + bx + c$ are consecutive integers?

7 Let a_1, a_2, ... be an infinite sequence of integers such that $0 \le a_k \le k$ for every positive integer k and such that

$$2017 = \sum_{k=1}^{\infty} a_k \cdot k!\,.$$

What is the value of the infinite series $\sum_{k=1}^{\infty} a_k$?

8 Let c be a complex number. Suppose there exist distinct complex numbers r, s, and t such that for every complex number z, we have

$$(z-r)(z-s)(z-t) = (z-cr)(z-cs)(z-ct).$$

Compute the number of distinct possible values of c.

9 Say that a positive integer n is *smooth* if $\frac{1}{n}$ has a terminating decimal expansion. (Note that 1 is smooth.) Compute the value of the infinite series

$$\sum_{n} \frac{1}{n^3}\,,$$

where n ranges over all smooth positive integers.

10 Let C be a cube. Let P, Q, and R be random vertices of C, chosen uniformly and independently from the set of vertices of C. (Note that P, Q, and R might be equal.) Compute the probability that some face of C contains P, Q, and R.

11 Let $S(N)$ be the number of 1's in the binary representation of an integer N, and let $D(N) = S(N+1) - S(N)$. Compute the sum of $D(N)$ over all N such that $1 \le N \le 2017$ and $D(N) < 0$.

12 Let S be the set of all real values of x with $0 < x < \pi/2$ such that $\sin x$, $\cos x$, and $\tan x$ form the side lengths (in some order) of a right triangle. Compute the sum of $\tan^2 x$ over all x in S.

13 A polynomial whose roots are all equal to each other is called a *unicorn*. Compute the number of distinct ordered triples (M, P, G), where M, P, G are complex numbers, such that the polynomials $z^3 + Mz^2 + Pz + G$ and $z^3 + Gz^2 + Pz + M$ are both unicorns.

14 A *permutation* of a finite set S is a one-to-one function from S to S. Given a permutation f of the set $\{1, 2, \ldots, 100\}$, define the *displacement* of f to be the sum

$$\sum_{i=1}^{100} |f(i) - i|\,.$$

How many permutations of $\{1, 2, \ldots, 100\}$ have displacement 4?

15 A restricted rook (RR) is a fictional chess piece that can move horizontally or vertically (like a rook), except that each move is restricted to a neighboring square (cell). If RR can only (with at most one exception) move up and to the right, how many possible distinct paths are there to move RR from the bottom left square to the top right square of a standard 8-by-8 chess board? Note that RR may visit some squares more than once. A path is the sequence of squares visited by RR on its way.

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16 Samantha is about to celebrate her sweet 16th birthday. To celebrate, she chooses a five-digit positive integer of the form SWEET, in which the two E's represent the same digit but otherwise the digits are distinct. (The leading digit S can't be 0.) How many such integers are divisible by 16?

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17 Circle ω_1 with radius 3 is inscribed in a strip S having border lines a and b. Circle ω_2 within S with radius 2 is tangent externally to circle ω_1 and is also tangent to line a. Circle ω_3 within S is tangent externally to both circles ω_1 and ω_2, and is also tangent to line b. Compute the radius of circle ω_3.

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18 Let x, y, and z be nonnegative integers that are less than or equal to 100. Suppose that $x+y+z$, $xy+z$, $x+yz$, and xyz are (in some order) four consecutive terms of an arithmetic sequence. Compute the number of such ordered triples (x, y, z).

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19 Up to similarity, there is a unique nondegenerate convex equilateral 13-gon whose internal angles have measures that are multiples of 20 degrees. Find it. Give your answer by listing the degree measures of its 13 *external* angles in clockwise or counterclockwise order. Start your list with the biggest external angle. You don't need to write the degree symbol $^\circ$.

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20 Compute the value of the sum

$$\sum_{k=1}^{11} \frac{\sin(2^{k+4}\pi/89)}{\sin(2^k\pi/89)}.$$
