First Brazilian Math Olympiad

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by Johann Peter Dirichlet

1 Show that if $a < b$ are in the interval $[0, \frac{\pi}{2}]$ then $a - \sin a < b - \sin b$. Is this true for $a < b$ in the interval $[\pi, \frac{3\pi}{2}]$?

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2 The remainder on dividing the polynomial $p(x)$ by $x^2 - (a+b)x + ab$ (where $a \neq b$) is $mx + n$. Find the coefficients m, n in terms of a, b. Find m, n for the case $p(x) = x^{200}$ divided by $x^2 - x - 2$ and show that they are integral.

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3 The vertex C of the triangle ABC is allowed to vary along a line parallel to AB. Find the locus of the orthocenter.

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4 Show that the number of positive integer solutions to $x_1 + 2^3x_2 + 3^3x_3 + \ldots + 10^3x_{10} = 3025$ (*) equals the number of non-negative integer solutions to the equation $y_1 + 2^3y_2 + 3^3y_3 + \ldots + 10^3y_{10} = 0$. Hence show that (*) has a unique solution in positive integers and find it.

5

- ABCD is a square with side 1. M is the midpoint of AB, and N is the midpoint of BC. The lines CM and DN meet at I. Find the area of the triangle CIN.
- The midpoints of the sides AB, BC, CD, DA of the parallelogram ABCD are M, N, P, Q respectively. Each midpoint is joined to the two vertices not on its side. Show that the area outside the resulting 8-pointed star is $\frac{2}{5}$ the area of the parallelogram.
- ABC is a triangle with CA = CB and centroid G. Show that the area of AGB is $\frac{1}{3}$ of the area of ABC.
- Is (ii) true for all convex quadrilaterals ABCD?
