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– Paper 1

1 There are n^2 students in a class. Each week all the students participate in a table quiz. Their teacher arranges them into n teams of n players each. For as many weeks as possible, this arrangement is done in such a way that any pair of students who were members of the same team one week are not in the same team in subsequent weeks. Prove that after at most $n+2$ weeks, it is necessary for some pair of students to have been members of the same team in at least two different weeks.

2 Determine all integers a for which the equation $x^2+axy+y^2=1$ has infinitely many distinct integer solutions x,y.

3 Points A,X,D lie on a line in this order, point B is on the plane such that $\angle ABX>120^\circ$, and point C is on the segment BX. Prove the inequality:
$2AD\geq \sqrt{3}(AB+BC+CD)$.

4 Consider the following one-person game played on the real line. During the game disks are piled at some of the integer points on the line. To perform a move in the game, the player chooses a point j at which at least two disks are piled and then takes two disks from the point j and places one of them at $j-1$ and one at $j+1$. Initially, $2n+1$ disks are placed at point 0. The player proceeds to perform moves as long as possible. Prove that after $\frac{1}{6}n(n+1)(2n+1)$ moves no further moves will be possible and that at this stage, one disks remains at each of the positions $-n,-n+1,...,0,...n$.

5 Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that for all real numbers x,y:
$xf(x)-yf(y)=(x-y)f(x+y)$.

– Paper 2

1 Prove that for every positive integer n,
$n^n\leq (n!)^2\leq \left(\frac{(n+1)(n+2)}{6}\right)^n$.

2 Let a,b,c be complex numbers. Prove that if all the roots of the equation $x^3+ax^2+bx+c=0$ are of module 1, then so are the roots of the equation $x^3+|a|x^2+|b|x+|c|=0$.

3 Let S be the square consisting of all pints (x,y) in the plane with $0\leq x,y\leq 1$. For each real number t with $0<t<1$, let C_t denote the set of all points $(x,y)\in S$ such that (x,y) is on or

above the line joining $(t, 0)$ to $(0, 1-t)$.
Prove that the points common to all C_t are those points in S that are on or above the curve $\sqrt{x} + \sqrt{y} = 1$.

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4 Points P, Q, R are given in the plane. It is known that there is a triangle ABC such that P is the midpoint of BC, Q the point on side CA with $\frac{CQ}{QA} = 2$, and R the point on side AB with $\frac{AR}{RB} = 2$. Determine with proof how the triangle ABC may be reconstructed from P, Q, R.

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5 For each integer n of the form $n = p_1p_2p_3p_4$, where p_1, p_2, p_3, p_4 are distinct primes, let $1 = d_1 < d_2 < ... < d_{15} < d_{16} = n$ be the divisors of n. Prove that if $n < 1995$, then $d_9 - d_8 \neq 22$.
