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– Paper 1

1 Find all real numbers x which satisfy: $\frac{x^2}{(x+1-\sqrt{x+1})^2} < \frac{x^2+3x+18}{(x+1)^2}$.

2 Show that there is a positive number in the Fibonacci sequence which is divisible by 1000.

3 If AD is the altitude, BE the angle bisector, and CF the median of a triangle ABC, prove that $AD, BE,$ and CF are concurrent if and only if:
$a^2(a-c)=(b^2-c^2)(a+c),$

where a,b,c are the lengths of the sides BC, CA, AB, respectively.

4 A 100×100 square floor consisting of 10000 squares is to be tiled by rectangular 1×3 tiles, fitting exactly over three squares of the floor. (a) If a 2×2 square is removed from the center of the floor, prove that the rest of the floor can be tiled with the available tiles. (b) If, instead, a 2×2 square is removed from the corner, prove that such a tiling is not possble.

5 The sequence $u_n, n=0,1,2,...$ is defined by $u_0=0, u_1=1$ and for each $n\geq 1$, u_{n+1} is the smallest positive integer greater than u_n such that $\{u_0,u_1,...,u_{n+1}\}$ contains no three elements in arithmetic progression. Find u_{100}.

– Paper 2

1 Solve the system of equations:
$y^2=(x+8)(x^2+2), y^2-(8+4x)y+(16+16x-5x^2)=0.$

2 A function $f:\mathbb{N}\to\mathbb{N}$ satisfies: (a) $f(ab)=f(a)f(b)$ whenever a and b are coprime; (b) $f(p+q)=f(p)+f(q)$ for all prime numbers p and q.
Prove that $f(2)=2, f(3)=3$ and $f(1999)=1999$.

3 The sum of positive real numbers a,b,c,d is 1. Prove that:
$\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+d}+\frac{d^2}{d+a}\geq\frac{1}{2},$

with equality if and only if $a=b=c=d=\frac{1}{4}$.

4 Find all positive integers m with the property that the fourth power of the number of (positive) divisors of m equals m.

5 A convex hexagon $ABCDEF$ satisfies $AB = BC, CD = DE, EF = FA$ and: $\angle ABC + \angle CDE + \angle EFA = 360^\circ$. Prove that the perpendiculars from A, C and E to FB, BD and DF respectively are concurrent.