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by laegolas, moldovan

–	Paper 1
1	Consider the set S of all numbers of the form $a(n)=n^2+n+1, n \in \mathbb{N}$. Show that the product $a(n)a(n+1)$ is in S for all $n \in \mathbb{N}$ and give an example of two elements s,t of S such that $s,t \notin S$.
2	Let $ABCDE$ be a regular pentagon of side length 1. Let F be the midpoint of AB and let G and H be the points on sides CD and DE respectively $\angle GFD = \angle HFD = 30^\circ$. Show that the triangle GFH is equilateral. A square of side a is inscribed in $\triangle GFH$ with one side of the square along GH. Prove that: $FG=t=\frac{2\cos 18^\circ \cos^2 36^\circ}{\cos 6^\circ}$ and $a=\frac{t\sqrt{3}}{2+\sqrt{3}}$.
3	Let $f(x)=5x^{13}+13x^5+9ax$. Find the least positive integer a such that 65 divides $f(x)$ for every integer x.
4	The sequence $a_1<a_2<...<a_M$ of real numbers is called a weak arithmetic progression of length M if there exists an arithmetic progression $x_0,x_1,...,x_M$ such that: $x_0 \le a_1 < x_1 \le a_2 < x_2 \le ... \le a_M < x_M$. (a) Prove that if $a_1<a_2<a_3$ then (a_1,a_2,a_3) is a weak arithmetic progression. (b) Prove that any subset of $\{0,1,2,...,999\}$ with at least 730 elements contains a weak arithmetic progression of length 10.
5	Consider all parabolas of the form $y=x^2+2px+q$ for $p,q \in \mathbb{R}$ which intersect the coordinate axes in three distinct points. For such p,q, denote by $C_{p,q}$ the circle through these three intersection points. Prove that all circles $C_{p,q}$ have a point in common.
–	Paper 2
1	Prove that if x,y are nonnegative real numbers with $x+y=2$, then: $x^2y^2(x^2+y^2) \le 2$.
2	In a cyclic quadrilateral $ABCD$, a,b,c,d are its side lengths, Q its area, and R its circumradius. Prove that: $R^2=\frac{(ab+cd)(ac+bd)(ad+bc)}{16Q^2}$. Deduce that $R \ge \frac{(abcd)^{\frac{3}{4}}}{Q\sqrt{2}}$ with equality if and only if $ABCD$ is a square.

3 For each positive integer n find all positive integers m for which there exist positive integers $x_1 < x_2 < ... < x_n$ with:
$\frac{1}{x_1} + \frac{2}{x_2} + ... + \frac{n}{x_n} = m.$

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4 Show that in each set of ten consecutive integers there is one that is coprime with each of the other integers. (For example, in the set $\{114, 115, ..., 123\}$ there are two such numbers: 119 and $121.$)

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5 Let $p(x) = a_0 + a_1x + ... + a_nx^n$ be a polynomial with nonnegative real coefficients. Suppose that $p(4) = 2$ and $p(16) = 8$. Prove that $p(8) \leq 4$ and find all such p with $p(8) = 4$.
