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–	Paper 1

1 1. (a) For which positive integers n, does 2n divide the sum of the first n positive integers?
(b) Determine, with proof, those positive integers n (if any) which have the property that 2n + 1 divides the sum of the first n positive integers.

2 Each of the players in a tennis tournament played one match against each of the others. If every player won at least one match, show that there is a group A; B; C of three players for which A beat B, B beat C and C beat A.

3 AB is a chord of length 6 of a circle centred at O and of radius 5. Let $PQRS$ denote the square inscribed in the sector OAB such that P is on the radius OA, S is on the radius OB and Q and R are points on the arc of the circle between A and B. Find the area of $PQRS$.

4 Prove that there are only two real numbers x such that

$$(x-1)(x-2)(x-3)(x-4)(x-5)(x-6)=720$$

5 Let $a,b\ge 0$. Prove that

$$\sqrt{2}\left(\sqrt{a(a+b)^3}+b\sqrt{a^2+b^2}\right)\le 3(a^2+b^2)$$

with equality if and only if $a=b$.

– Paper 2

1 Determine all pairs of prime numbers (p,q), with $2\le p,q<100$, such that $p+6, p+10, q+4, q+10$ and $p+q+1$ are all prime numbers.

2 A and B are distinct points on a circle T. C is a point distinct from B such that $|AB|=|AC|$, and such that BC is tangent to T at B. Suppose that the bisector of $\angle ABC$ meets AC at a point D inside T. Show that $\angle ABC>72^\circ$.

3 Suppose n is an integer ≥ 2. Determine the first digit after the decimal point in the decimal expansion of the number

$$\sqrt[3]{n^3+2n^2+n}$$

4 Define the function m of the three real variables x, y, z by $m(x,y,z) = \max(x^2,y^2,z^2)$, x, y, z R.

Determine, with proof, the minimum value of m if x,y,z vary in R subject to the following restrictions:

$x + y + z = 0$, $x^2 + y^2 + z^2 = 1$.

5 Suppose p, q are distinct primes and S is a subset of $\{1, 2, \ldots, p-1\}$. Let $N(S)$ denote the number of solutions to the equation

$$\sum_{i=1}^{q} x_i \equiv 0 \mod p$$

where $x_i \in S$, $i = 1, 2, \ldots, q$. Prove that $N(S)$ is a multiple of q.